A profile of Irish second year post-primary students' knowledge of initial algebra

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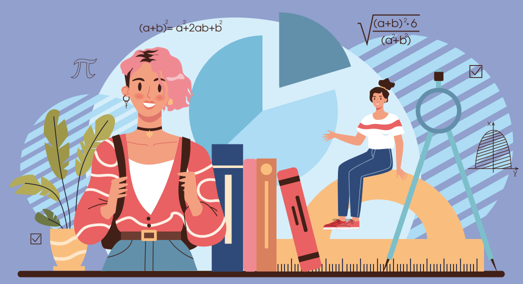
Foreword



I am pleased and honoured to write a Foreword to this important report, *A profile of Irish second year post-primary students' knowledge of initial algebra* by Dr Aoife O'Brien and Dr Máire Ní Ríordáin based on Dr O'Brien's doctoral thesis. As a lifelong mathematics educator and researcher who is passionate about improving all aspects of mathematics/STEM teaching and learning through evidence-based research, I am delighted to be associated with the authors' major empirical study on post-primary school algebra in Ireland.

My first encounter with the author, Dr Aoife O'Brien, was at a mathematics education conference (ICME-13). Her supervisor and mentor, Dr Máire Ní Ríordáin, a wonderful colleague and friend, introduced us. Dr O'Brien was then a lecturer in mathematics at Galway-Mayo Institute of Technology (GMIT) and continues in this post today. She had just started her academic journey as a doctoral student with my colleague, researching the learning and teaching of algebra in Irish post-primary schools. I was excited when I learned the exact nature of her project. It appeared to intersect and dovetail with two doctoral studies that I supervised at EPI*STEM, The National Centre for STEM Education at the University of Limerick, one completed and the other ongoing at the time. Subsequently, I stayed in touch with her project and was delighted to serve on the author's expert panel and offer advice on the development of her algebra screener, a wonderful tool for mathematics teachers.

Competency in algebra and algebraic thinking are core concerns in all reform curricula internationally including the new mathematics curriculum in Irish post-primary schools. Algebra is challenging for students to learn and for teachers to teach at primary and post-primary level, and this reality is universally recognised. This state of affairs is underlined by persistent underperformance in algebra in state examinations as reported by the ERC, DES, Chief Examiner's reports, and international surveys where Irish schoolchildren have participated. Details of these and other research are contained in the report's extensive bibliography.

Such reports are excellent and necessary in what they do to highlight issues with algebra and mathematics education generally; however, they do not offer solutions for underperformance. Unfortunately, there is no magic wand to conjure up a solution nor is it likely that there is a single solution. However, evidence-based empirical research such as that undertaken by the authors and reported here is essential if we are to understand obstacles to progress and chart pathways to improved performance. We are very fortunate that we have in Ireland at this time, a vibrant community of expert researchers in Mathematics/STEM education capable of rising to the challenges ahead, ranking high among them the report's authors.

This study is unique in the Irish context in what it does and how it does it. The authors address a gap in our knowledge of Irish students' mathematical knowledge at the transition from arithmetic to initial algebra (that occurs in year 2 of their post-primary education) particularly as regards the strengths and weaknesses in their mathematical profile. The authors apply and exploit relevant international research to guide their work, and in particular to develop a customised algebra test (or screener) validated for the target audience to collect empirical data in a large-scale study of Irish second year students' knowledge of initial algebra.

Clearly, the goal is to improve students' mathematics education through better algebra learning and teaching. The authors appreciate the importance of this goal and the significance of algebra for its success. The focus on students' strengths and weaknesses establishes an important baseline from which to start the process of building pathways to better understanding and improvements. The report adds new evidence and insights in this area of study and is well balanced in that it invites stakeholders to consider both *strengths* and *weaknesses* in their efforts to support teachers and students in this core area of the mathematics curriculum.

There is much to be gained by improving mathematics education in schools since mathematics underpins all the STEM and numerate disciplines. The stakes are very high and the rewards for individuals and society are great: significant improvement in algebra learning and teaching will lead to better outcomes in their mathematics education for post-primary students, better preparation for STEM disciplines and other numerate disciplines in Higher Education (HE), and better numeracy in the wider population. The expected payoff for better outcomes in these areas is better economic outcomes and national prosperity.

The importance of algebra in children's mathematics education and the nature and quality of the research reported here make this a very important report for the advancement of Mathematics/STEM education in Ireland. This report is essential reading for policy makers and all other stakeholders in Irish education. The authors offer new tools, evidence and insights for the advancement of algebra learning and teaching in school mathematics that will affect the quality of students' algebra and overall mathematics education and the preparation of their mathematics teachers. By focussing on algebra learning and teaching, the authors chart a course for the improvement of mathematics education in Irish Junior Cycle schools, and do this in an evidence-based way. This a very insightful and far-reaching approach nationally and internationally since improvements in core areas of the STEM disciplines will contribute to better STEM education outcomes.

The authors are to be commended for their work and making their findings accessible to a wider readership nationally and internationally in this report, while catering for a primary readership of practitioners, teacher educators and policy makers, and mathematics/STEM education researchers.

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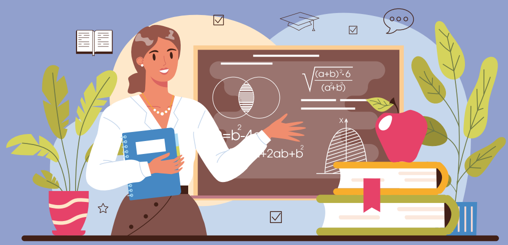
Contents

List of Figures and Tables

Executive Summary



This research focuses on initial algebra as students transition from arithmetic to algebra, an area of mathematics education known for its difficulties (Demonty *et al.*, 2018; Kieran, 2007). Algebra is identified by many as the language of mathematics and considered a prerequisite for further study in mathematics in many disciplines such as science, technology, and engineering (Gavin & Sheffield, 2015; Stacey & Chick, 2004). It has been argued that algebra is not only the gatekeeper for higher education but also for citizenship.

Objectives of the Study

This report provides a profile of Irish second years students' knowledge of initial algebra. Students need to develop knowledge, skills, and abilities across a wide range of content areas to ensure success with initial algebra. The prerequisite content areas include ratio and proportional reasoning, fractions, decimals, percentages, integers, exponents, order of operations and equality. The algebra content areas include variables, expressions, equations, functions, and patterns (Bush & Karp, 2013; Warren & Cooper, 2008).

The central objective of the study was to develop a profile of Irish second year students' knowledge of initial algebra and from this:

1.	Identify students' strengths and weaknesses in the pertinent content areas for initial algebra.
2.	Identify prevalent errors and misconceptions that hinder students' understanding and progression with initial algebra.
3.	Provide the empirical evidence to help mathematics teachers effectively guide their instruction given the limited time available in the classroom.
4.	Develop key recommendations to support mathematics teachers and researchers in the area of initial algebra.

Evidence has emerged, through various government reports, research, and international testing, that students in Ireland, like their international counterparts, are experiencing difficulties with algebra (Chief Examiner, 2015a; Shiel & Kelleher, 2017). This study provides empirical evidence based on the assessment of over 500 second year post-primary students during the academic year 2016-2017, as part of a PhD study. This report provides a summary of the results from this assessment and identifies key content areas with which our students are struggling. This empirical evidence has heretofore been unavailable.

Study Design and Implementation

A two-stage cluster sample design was employed as is common in educational studies as it is practical and allows for analysis at more than one level of data aggregation, that is between student or between classes or indeed both if required (Ross, 2005). This type of design involves the selection of schools (first stage) for inclusion in the study, followed by the selection of a cluster (class) of students within the school (second stage). A non-probability sampling technique known as snowball sampling where one participant (teacher/school principal) recommends at least one other participant who in turn recommends another was employed in this study (Creswell, 2012). The initial teachers were recruited through social media channels and personal and organisational contacts of the researchers. The students were accessed solely through their teachers. Since a non-probability sample method was used the final sample may not entirely represent the population and therefore the generalisability of the results may be impacted.

The final sample comprised 19 post-primary schools, 29 teachers/classes, which resulted in a sample of 667 students. Of the 19 schools, 11 (57.9%) were secondary schools, 7 (36.8%) were vocational schools and 1 (5.3%) was a community/comprehensive school. All schools were English medium schools and 3 (15.8%) carried a DEIS indicator. Due to absences on the day of administration and lack of consent/assent 555 students' data were included in October 2016 and 476 in April 2017. Of the 555 students assessed in October 305 (55.0%) were male, 248 (44.7%) were female and 2 (0.4%) preferred not to say. In April 264 (55.5%) were male and 212 (44.5%) were female.

The assessment utilised, known as a screener, was specifically developed, and validated for Irish second year post-primary students as a part of this study. Ethical approval was sought and granted for this research through the National University of Ireland, Galway. Students were assessed in their own classrooms supervised by their own mathematics teacher in early October 2016 at the start of the academic year and again in April 2017 towards the end of the academic year.

Key Findings

The empirical results reported here are the first of their kind for Irish second year post-primary students and provide valuable information for researchers and educators in the Irish context. They also provide important evidence in the international context that algebraic errors are common to students of all nationalities exposed to different curricula worldwide. The following provides a summary of key findings and their importance for mathematics teaching and learning.

Students' Strengths:

Students who can interpret the equal sign correctly and see it as a relational symbol have more flexibility when working with equations and therefore a sound understanding of equality is essential to succeed with algebra. Those who misinterpret the meaning of the sign, taking it to mean 'perform a calculation' that is viewing it as an operational symbol will struggle to work with equations (Stephens *et al.*, 2013). Two items on the screener assessed equality, and these were the best answered items with over 80% of students answering both items correctly. Therefore, the important content area of equality is well understood by the sample and shows that the effort at primary level dedicated to this important concept has been worthwhile (NCCA, 1999).

Proportional reasoning is regarded as one of the major components of formal thought and once achieved this form of thinking can assist with learning throughout all disciplines of science, mathematics, and life (Bush & Karp, 2013). Students do not always develop proportional reasoning skills naturally, and often mistakenly apply additive thinking to problems that require proportional reasoning (Hilton *et al.* 2013). A visual assessment task was used to assess proportional reasoning and over 70% of students answered correctly demonstrating that the majority have a good understanding of this key content area. Proportional reasoning is a key aspect of numeracy covered at junior cycle (NCCA, 2017) and these results demonstrate that most of our students understand the concept.

When working with numbers students often learn a particular procedure for dealing with a type of number, for example using the common denominator when adding fractions. Therefore they rely on memory when trying to deal with these numbers (Bush & Karp, 2013). The lack of understanding of different number formats and associate procedures leads to an inability to apply the rules correctly when working with variables in algebra. Evidence of students struggling with fractions and decimals has emerged in the results of this study and these are discussed below. However, most students (over 64%) were able to compare 40% of €400 and 0.75 of €200 correctly without the use of a calculator. Being able to compare and order numbers are important skills and required for applying rules correctly when working with variables. It also enables a student to assess if a solution to an equation or inequality is reasonable (Bottoms, 2003).

The distributive property is fundamental to students' ability in algebra, given that it is used frequently in the transformation of expressions in both simplifying and factorising. In algebra finding equivalent expressions is frequently required, for example an expression being factorised $3x + 6 = 3(x + 2)$. This manipulation requires an underlying sense of the properties of numbers (Bush & Karp, 2013). Procedural proficiency with the distributive property when dealing with an expression was evidenced by most students (over 64%). Traditionally students are given the opportunity to practise the distributive property with repeated drill exercises of the form 3(x + 2), that is multiplying out brackets, which allows students to become proficient in applying the property without acquiring a full understanding of it. That is students are fluent in the transformative rules and symbol manipulation of algebra, however they view these as superficial movement of symbols without an actual understanding of the structural properties (Kaput, 1989). Without this true understanding, errors occur in the students' work as they reach more advanced algebraic expressions. Evidence of a lack of this deeper understanding of the distributive property emerged from the responses to other items on the screener specifically in relation to exponents which is discussed below. Many researchers believe that allowing students to investigate the properties of numbers will assist students in learning, retaining knowledge and relational understanding, which in turn will create a strong foundation for algebra (Bush & Karp, 2013).

The ability to see patterns, explain them and represent them with algebraic symbolism is important for mathematical problem solving (Ayalon *et al.*, 2015). When working with patterns the relationship which lies between the pattern and its position in a functional relationship must be identified, and to do this an expression or formula must be created using variables. In doing this a context for the use of variables is set for students, assisting their understanding of a variable as a varying quantity rather than a specific unknown (TaniÐli & Köse, 2011). In this study students were asked to complete a table for the number of sides in the perimeter of a stacked hexagon pattern. This was well answered with over 70% of students completing the table correctly. Identifying the correct description and formula to represent the pattern however was a less well answered part of the task (~50% and ~30% respectively). This points to lack of understanding of variables and the ability to form an expression.

Forming algebraic expressions is a notoriously difficult task, whereby the translation from word problem to expression often results in errors (Bush & Karp, 2013). The results of this study show that when asked to write an expression for the perimeter of an equilateral triangle of side *c,* that over 70% were correctly able to write *3c*. When asked to write the expression for a perimeter of a pentagon with 4 equal sides of *h* and one side *t,* just over half of students succeeded. The most difficult task showing an open sided shape with *n* sides of length 2 was answered correctly by very few. Often students' struggles with algebraic expression result from misconceptions with variables and this lack of understanding of variables is evident in the results (Bush & Karp, 2013). Further issues with algebraic expressions arise when students fail to simplify properly in that they do not add and subtract like terms, for example a common error is to simplify 2x + 4 to 6x (Kieran, 1992). Another error is to detach terms (which may or may not contain a variable) from the operations (Jupri *et al.*, 2014; Linchevski, 1995). Furthermore, believing that the sign 'belongs' to the preceding term rather than the term after leads to simplifying expressions such as 2 - 3x + 2x to 2 - 5x. The task assessing simplification of an algebraic expression was answered correctly by approximately half of the sample.

Students' Weaknesses:

The results of this study show that most of the sample struggled with the pertinent content areas of fractions, decimal number magnitude, exponents, integers, order of operations, variables, forming expressions, and forming and solving equations. An understanding of these key prerequisite content areas is essential for success with algebra and the results here point to why our students are struggling with algebra overall (Bush & Karp, 2013).

Siegler *et al.* (2012) conducted a study using longitudinal data from the UK (1980-1986) and USA (1997-2002). The aim was to identify long term predictors of post-primary students' ability in algebra. It revealed that a student's knowledge of fractions and division uniquely predicted a student's knowledge of algebra and overall mathematical ability five or six years later. In Ireland, a lack of knowledge of fractions on entering post-primary school is reported as impacting negatively on algebra performance (Shiel & Kelleher, 2017). Four items on the screener assessed fractions and only one of these items were answered correctly by more than 50% of the sample at both administrations. The remainder of the items were answered correctly by less than 1 in 2 students. Items assessing equivalent fractions and multiplying fractions were answered correctly by approximately 2 in 5 students, while the item assessing relational knowledge of fractions was answered correctly by only 1 in 5. Fraction knowledge is an important prerequisite for the study of algebra and there are numerous common errors encountered in student workings both nationally and internationally (Booth *et al.*, 2014; Bush & Karp, 2013).

A study by DeWolf *et al.* (2016) aimed to pinpoint what aspect of fraction knowledge was best at predicting algebra performance. They concluded that an understanding of decimal magnitudes when assessed by a number line task and a relational understanding of fractions are strong predictors of algebra performance. A number line task was used to assess decimal number magnitude where students were asked to place a hatch mark on the number line for a given decimal number. More than half of the students in the sample (55%) could not place the hatch mark within plus or minus 10% of the correct location which indicates difficulties with understanding decimal number magnitude. Items assessing relational fraction knowledge, were answered correctly by only 1 in 5 of students. Our students are struggling with these key prerequisite areas for success with algebra.

An understanding of exponents is required throughout algebra in both the transformational skills and the generational global/meta level skills (Bottoms, 2003; Bush & Karp, 2013). Transformational skills include factorising, multiplying, dividing, and simplifying algebraic expressions all require a knowledge of exponents. Two items assessed exponents, and these were among the worst answered items of all. Approximately only one in ten students were correctly able to square an expression. The more straightforward item on asking students to square x^5 was answered correctly by approximately one in three students only. Exponential notation consists of two parts, a base and exponent and it is a way of expressing repeated multiplication. The ability to understand and work with exponents requires understanding the notation, the meaning and properties of exponents and it is known to be a difficult mathematical topic for students of all ages (Ulusoy, 2019).

Knowing the order in which to carry out arithmetic operations is a fundamental skill required for algebra (Bottoms, 2003). There are many common misconceptions when dealing with the order of operations, the most common of which is carrying out operations in order from left to right (Booth & Carlton Johnson, 1984). One item assessed order of operations asking students to evaluate 13 - 3 x 4 + 2. Only one in three students approximately were able to evaluate this correctly as 3. The most common error was indeed working left to right, with approximately one in three students doing this. Additionally, just less than one in five students answered -1, believing that addition supersedes subtraction. This may have to do with a lack of understanding of integers or it may be as a result of having learned order of operations through a mnemonic (Schwartzman, 1996) e.g., BOMDAS/BIRDMAS.

Kieran (1992) asserts that many misconceptions and common errors in algebra are generally rooted in the meaning of symbols or the letters used. Much research has been conducted into students' difficulties in working with algebraic variables and it is identified as a key area for student misconceptions (Asquith *et al.*, 2007; Booth *et al.*, 2014; Bush & Karp, 2013; Jupri *et al.*, 2014; Perso, 1991). An essential element of algebraic thinking is a rich understanding of variables (Hunter, 2010). Common misconceptions with variables include viewing variables as labels, and the belief that a variable is just a missing value rather than varying values (Asquith *et al.*, 2007; Bush & Karp, 2013). Evidence of second year students incorrectly understanding a variable as a label has emerged in this study. One item designed by Küchemann (1981), to identify a students' understanding of a variable was answered correctly by approximately one in four students. Two in five students understood the expression 8b + 6m to mean "8 books and 6 magazines" despite being told that b stood for the number of books and m stood for the number of magazines. When a student is at the stage of understanding a variable as a label only, this results in further errors with algebraic expressions (Küchemann, 1978). In the original study by Küchemann (1981), it was reported that 39% of 14-year-olds made this error. A study from almost twenty years ago in the US used this item (modified to 4 cakes and 3 brownies) and found that 37% of grade 7 students and 27% of grade 8 students made the same error (equivalent to first and second year post primary in Ireland). It appears that despite the changes in pedagogy and approaches to teaching algebra internationally and in Ireland, the proportion of students making this error has increased over the decades (Kieran *et al.*, 2016; Prendergast & Treacy, 2017).

Algebra has been viewed by many for centuries as the science of equation solving (Kieran, 2004). An ability to work with variables, write algebraic expressions, and to form and solve equations is the essence of success for initial algebra as it is fundamental in preparing for more advanced algebraic concepts (Capraro & Joffrion, 2006). There are several items on the screener that assess knowledge of equations. The first of these asks students to solve 4 - x = 5 and less than 1 in 3 students could solve this equation at the beginning of second year. The main error in attempting to solve the equation arises from the incorrect application of the addition inverse, possibly due to the minus in front of the x, whereby students may be confused as to whether the negative sign means subtraction or is part of a negative number (Vlassis, 2008). This lack of understanding is further evidenced on the item asking for the next step to solve 5z = 30 (Chung & Delacruz, 2014). The most prevalent error was students incorrectly choosing z = 30 - 5, the incorrect application of subtraction as the inverse to multiplication in solving the equation.

Solving linear equations is difficult with up to fifty concepts involved (Chung & Delacruz, 2014) and it was not possible to investigate all within the scope of this study. The results of various items show that the minus sign is problematic for some students where they are possibly viewing it at the subtrahend level of subtraction only (Vlassis, 2008). This is evident in the responses to the item on order of operations, where some students mistakenly believe that addition supersedes subtraction as mentioned. It is also evidenced in the item which asks students to solve 4 - x = 5, with some students dropping or ignoring the minus sign when solving this equation. Issues with the minus sign are also evident from another item assessing equations which asked students to identify the next correct step in solving 7h - (3h - 2) = 38. The errors made on this item identify that approximately two in five students do not recognise the minus sign as an operational signifier (Vlassis, 2008).

Key Recommendations

Irish students struggle with initial algebra as much as their international counterparts. Evidence presented here shows many strengths as well as common errors and misconceptions recognised in the literature, and it provides a much greater depth of understanding of Irish post-primary students' difficulties than was previously known. It is hoped that this research will serve all stakeholders in mathematics education in understanding the strengths and weaknesses of students as they begin to grasp algebraic concepts. The information produced here can be used to determine how best to support teachers and students in this area of mathematics education which is well documented as difficult to teach and learn (Demonty *et al.*, 2018; Kieran, 2007).

The key recommendations based on the findings from this research include:

- The development of evidence-based teaching interventions in key content areas to help remediate the common errors and misconceptions held by our students.
- The appointment of specialist mathematics educators in both primary and post-primary schools to support existing teachers and instruction.
- The development and validation of a suite of cognitive diagnostic assessments. These assessments would help educators and researchers fully understand students' misconceptions and thinking in more depth, leading to more targeted use of appropriate interventions (Groß *et al.*, 2016).
- Professional development for mathematics educators where required and the opportunity to work collaboratively. Discussion around the common errors identified in this study among teachers can allow for tailored specific plans for instruction to help remediate these errors.
- The consideration of these findings at policy level, for both primary and junior cycle levels. It is recommended that a bridging course between primary and post-primary relating to prerequisite algebra be developed to support learners in the transition from arithmetic to algebraic thinking.

The information provided in this research report can help mathematics teachers and researchers from primary level through to third level by allowing them to focus on strengths and key problem areas relating to algebra. By being more aware of the misconceptions and areas of mathematics with which their students struggle teachers can plan instruction accordingly. "If one is truly to succeed in leading a person to a specific place, one must first and foremost take care to find him where he is and begin there" (Kierkegaard, 1962, p. 45).

1 Introduction

This report presents evidence of second year post-primary students' knowledge of initial algebra. Much research and effort has been focused on the area of algebraic thinking and initial algebra to improve students' engagement and attainment with the subject (Kieran *et al.*, 2016). Despite this, students internationally continue to struggle (Demonty *et al.*, 2018). Considerable research has taken place to understand students' difficulties with mathematics and ways in which to help them, however, despite this, there is no clear evidence that this effort has had a significant impact on attainment (Hodgen *et al.*, 2009). Additionally, there have been numerous studies internationally focusing on algebra to help understand and remediate students' difficulty (Booth *et al.*, 2014). At a national level, new methods and modes of teaching algebra have been implemented (Prendergast & Treacy, 2017). Evidence has emerged, through various government reports, research, and international testing, of students in Ireland experiencing difficulties with algebra (Chief Examiner, 2015a; Shiel & Kelleher, 2017). Additionally, Irish post-primary teachers have highlighted that our students are struggling with many of the prerequisite content areas for example, fractions (Shiel & Kelleher, 2017). The analysis presented in this report provides detailed insight into the specific difficulties Irish second year post-primary students are encountering with algebra. Previously, there has been little to no empirical evidence available on second year students' knowledge of algebra in Ireland.

2 Aim of the study

The main aim of this research is to establish a profile of what second year students in Ireland know about algebra, specifically prerequisite and initial algebra concepts. The evidence presented in this report is based on a pen-and-paper algebra assessment completed by over five hundred second-year students during the academic year 2016/2017. The assessment utilised, known as a screener, was specifically developed, and validated for Irish second year post-primary students, as part of this study.

3 Objectives

The objectives of this study were as follows:

1.	To investigate students' knowledge of initial algebra and identify their strengths and weaknesses in the pertinent content areas.
2.	To identify prevalent errors and misconceptions that hinder students understanding and progression with initial algebra.
3.	To provide the empirical evidence to help mathematics educators effectively guide their instruction given their limited time in the mathematics classroom.
4.	To develop key recommendations to support mathematics educators and researchers in the area of initial algebra.

4 The Irish Context

The Mathematics problem at third level in Ireland is well documented and Irish third level institutions are now encountering students entering science and technology-based programs with inferior mathematical skills when compared with students a decade ago (Prendergast & Treacy, 2017; Treacy & Faulkner, 2015). This is consistent with international trends (Treacy & Faulkner, 2015). A recent Higher Education Authority (HEA) report highlights the requirement for funding for "Maths enabling courses" and pre-maths courses for all third level institutes (Liston *et al.*, 2018). Students are leaving second level education underprepared for tertiary level mathematics.

The problems manifesting at third level are also in evidence at post-primary level. The Chief Examiner's (2015) reports for mathematics at both Junior and Leaving Certificate level were issued to provide information and analysis of students' performance in their examinations and to assist teachers with ongoing delivery and improvement of mathematics (Chief Examiner, 2015a, 2015b). The academic year 2015 was the first year in which all candidates for Junior Certificate examination were assessed entirely on the new syllabus; however, not all candidates at Leaving Certificate had been introduced to all elements of the new syllabus due to the phased implementation (Chief Examiner, 2015b).

There are many positives to be garnered from the reports, namely the greater proportion of students taking mathematics at higher-level and their achievement overall at same. Policy decisions were made to increase the uptake of higher-level mathematics, as it had the lowest rate of uptake at higher level over any other subject at both junior and senior cycles (Prendergast, 2011). Both reports commented on the change of content for post-primary mathematics education. The highest achieving students demonstrated good problem-solving abilities and use of all the studied strands of mathematics to tackle a problem (Chief Examiner, 2015b). However, the conclusions in both reports highlight issues for concern. At Leaving Certificate level it is noted

> *"The overall performance of some higher-level candidates with respect to their ability to apply basic skills appropriately and accurately is a cause for concern. It is clear that the proportion of the candidature for whom this is a significant difficulty has increased since 2011." (Chief Examiner, 2015b, p. 27)*

Furthermore, it is noted that ordinary level candidates display a worrying lack of competence in algebra and in particular algebraic manipulation (Chief Examiner, 2015b). At Junior Certificate level the same concerns are expressed about candidates at all three levels which "will undoubtedly cause problems for these candidates in Leaving Certificate Mathematics" (Chief Examiner, 2015a, p. 31).

A review by the Educational Research Centre (ERC) on behalf of the National Council for Curriculum and Assessment (NCCA) entitled "An Evaluation of the Impact of Project Maths on the Performance of Students in Junior Cycle Mathematics" provides further evidence of students' performance on algebra (Shiel & Kelleher, 2017). The review included interviews with 76 post-primary teachers as part of a Focus Group, who overall were happy with the current curriculum for algebra and junior cycle. It is noted that the introduction of patterns and the links between algebra and other aspects of the syllabus is working well. However, the teachers identified algebra as the most challenging area of mathematics for students and four common themes emerged in relation to algebra at junior cycle. First is that on entering post-primary students are displaying a lack of proficiency with fundamental mathematical skills. A lack of knowledge of fractions and integers in first-year post-primary is impacting on students' ability to progress with algebra. Secondly, the group were almost unanimous in their view that a lack of instructional time was hindering their efforts to teach algebra effectively at junior cycle. Teachers report that the time to revise and practise basic algebra skills is not available. Thirdly, it is reported that students can achieve high grades in their junior certificate examination by doing well in statistics and probability but by having little knowledge of algebra. This allows a student to progress to higher level mathematics at senior cycle, however the lack of proficiency in algebra then hinders a student from progressing with the subject overall. Finally, issues relating to better able students not being catered for emerged. The change in syllabus and initiatives to improve the uptake of mathematics at higher level, have resulted in large numbers in higher level classes with a greater range of abilities. Teachers are aware that differentiated instruction can help deal with this, but large class sizes and lack of instructional time are limiting their efforts (Shiel & Kelleher, 2017).

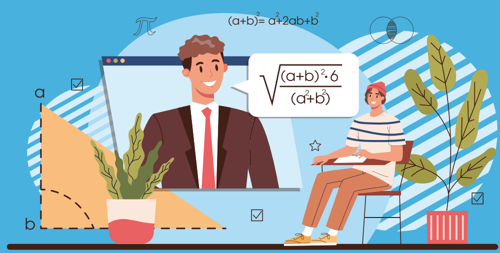
It is clear from the conclusions of both Chief Examiner's reports and that of the ERC that many students continue to struggle with algebra at junior cycle and further research is required to improve Irish students' attainment in this topic (Chief Examiner, 2015a, 2015b; Shiel & Kelleher, 2017). Research focusing on algebra education at the primary to post-primary level indicates that a focus on student difficulties together with the implementation of supportive teaching and learning environments can improve students' ability with algebra (Cai *et al.*, 2011; Demonty *et al.*, 2018; Kieran, 2007). The Chief Examiners' reports highlight students' difficulty with algebra, however this report is based on their Junior Certificate examination, after three years post-primary education have been completed. There is a clear need to establish students' difficulties at an earlier stage to assist teachers in identifying students' needs within the classroom environment and thereby improving instruction on an ongoing basis (Booth *et al.*, 2014).

In trying to establish a profile of students' knowledge of algebra it is appropriate to look at the results of Irish students on international tests. The Programme for International Student Assessment (PISA) and Trends in International Mathematics and Science Study (TIMSS) reports and results were analysed over the last twenty years to build a profile of Irish students' ability *vis a vis* their international counterparts. Ireland has participated in PISA, which takes place every three years, since inception in 2000. PISA aims to evaluate worldwide education systems by assessing 15-year-old students on how they apply their school learning to real world problems (OECD n.d.). Students are tested in the key subjects of Reading, Mathematics and Science and in addition to this, students are tested on an innovative domain in each round of testing, for example problem solving in 2015 (OECD n.d.).

In mathematics, generally Ireland performs in line with the overall average of approximately ninety participating countries, apart from in 2009 when Ireland scored an average 487.1, 12.3 marks below the average which was statistically significant. It should be noted that the scores on the PISA assessment are standardised with a mean of 500 and standard deviation of 100. However, in 2012, 2015 and 2018 Ireland performed better than the OECD average. It is not possible to establish performance on algebra specifically from the results of PISA as these assessments incorporate three key processes, *formulating, employing, and interpreting,* in the context of problem solving rather than assessing specific domains of mathematics (McKeown *et al.*, 2019). Ireland also participates in TIMSS, which provides information on students' performance on mathematics and science subjects. TIMSS data gets collected every four years since commencing in 1995. Students in grades four (fourth class primary in Ireland) and eight (second year post-primary in Ireland) are included in the study (National Center for Education Statistics, 2019). The average age of a student in Ireland participating in TIMMS at grade eight in 2015 was 14.4 years in line with the international average of 14.3 years. The results from TIMSS are in line with the PISA results with Ireland performing above average at both grade levels. However, although Ireland is performing well it is noted by Clerkin *et al.* (2016, p. 39) that "Second Year students displayed relative strengths on the Number (+21 points) and Data & Chance (+10 points) content domains, and relative weaknesses on Algebra (-22 points) and Geometry (-20 points)". These results concur with the findings by the ERC focus group, that overall performance in mathematics is boosted by students' knowledge of statistics and probability (Data & Chance), while students' ability with algebra lags (Shiel & Kelleher, 2017).

Algebra is the language of mathematics and a fundamental skill required for almost all STEM based and many other third level courses (Grønmo, 2018). There is a large body of research in adolescent difficulties with initial algebra and the consequences of these difficulties (Blanton *et al.*, 2015). Algebra is seen as a pivotal subject in the mathematics curriculum and research into the teaching and learning of algebra is seen as high priority (Huntley *et al.*, 2007). Despite the changes made to the Irish syllabus and teaching methods, problems with algebra persist. This is evidenced in the Chief Examiner's (2015) report which states students need to "gain comfort and accuracy in the basic skills of computation, algebraic manipulation and calculus" (p. 29). The evidence produced in this report will support teachers and schools by presenting more detail on the exact difficulties their students are encountering.

5 Definition of Initial Algebra



Initial algebra is when students are transitioning from arithmetic to algebra, and it is a crucial phase in algebra education (Van Amerom, 2002). Algebra is a formal symbolic language, composed of strings of symbols, which may form a 'sentence' within the language when constructed according to the syntax of that language. The language of algebra has five major aspects: (i) unknowns, (ii) formulas, (ii) generalised patterns, (iv) placeholders, and (iv) relationships (Usiskin, 1999). Initial algebra is an area of mathematics education that has received much focus and attention since the late 1980s when the concept of "*algebraic thinking*" was introduced (Kieran, 2004). It is believed that students should be introduced to the concepts of algebra much earlier in their education, from age 6, rather than the traditional view of introducing students to algebra after arithmetic at age approximately 12 (Kieran *et al.*, 2016).

There are two established frameworks for algebraic thinking in the literature: one proposed by Kieran (2004) and the other by Kaput *et al.*, (2008). Kieran (2004) sets out a model for school algebra which outlines three activities that learners must participate in:

1.	Generational activities: these involve the creating of expressions and equations, which involves becoming acquainted with some unknown that is the variable and the concept of equation solution. Most meaning-based learning for algebra occurs during these activities.
2.	Transformational activities: these are the "rule based" activities, where students learn how to manipulate algebraic symbols. Mastering these activities allows for the manipulation of an expression or equation while maintaining equivalence.
3.	Global/Meta Level activities – these can be described as problem solving and modelling activities. Algebra is used as a tool to solve and much of the sense making and real-life context of the mathematics is learned during these activities.

Generational activities are central to algebra and the theme of "generalisation" is to the fore in much of the research into school algebra, including generalising related to patterned activity and generalising related to properties of operations and numerical structure (Kieran *et al.*, 2016). Research conducted by Blanton and Kaput (2004) shows that children as young as the age of 5 can engage in pattern activities which allow them to engage in co-variational thinking and hence functional thinking. Generalising from numerical and geometric patterns had gained much research attention given children's ability to engage in these tasks (Bourke & Stacey, 1988; Küchemann, 2010; Lee *et al.*, 2011; Mason, 1996; Warren & Cooper, 2008). The use of patterns to identify mathematical change, relationships and to develop functional thinking is not intended to introduce students to formal algebra but rather to assist with their algebraic thinking (Kieran *et al.*, 2016). Students are expected to use natural language to give a general rule to describe the changing pattern but are not yet expected to formally represent functions with algebraic symbols. This method of engaging learners is known as the functions-based approach to algebra and was introduced to all post-primary schools in Ireland in September 2011 (Prendergast & Treacy, 2017).

Another conceptual approach for algebra is based on the work of Kaput *et al.* (2008). Its core aspects are "making and expressing generalizations in increasingly formal and conventional symbol systems and acting on symbols within an organized symbolic system through an established syntax" (Blanton *et al.*, 2019 p. 1934). These two core aspects occur across three content strands (i) algebra as the study of structures and systems that arise in arithmetic, (ii) algebra as the study of functions, relations and joint variation, and (iii) algebra as a cluster of modelling languages (Kaput *et al.*, 2008, p. 11). In comparing the two models, the generational activities from Kieran's model aligns with Kaput's first core aspect (although it is not equivalent), while the transformational activities align with the second core aspect. The global/meta activities from Kieran's model then align with the second and third content strands of Kaput's model (Hodgen *et al.*, 2018).

Both Kaput's and Kieran's models of algebraic thinking align closely with the junior cycle syllabus. Research by Blanton *et al.* (2018) to develop a framework for algebra organises Kaput's two core aspects into the following three areas which was adopted as the conceptual model for initial algebra in this study.

i. Generalised arithmetic: this includes generalising, justifying, representing, and reasoning with arithmetic relations which include the properties of numbers.

ii. Equivalence, expressions, equations, and inequalities: which includes the development of a relational understanding of the equivalence sign, together with generalising, justifying, representing, and reasoning with equivalence, expressions, equations, and inequalities.

iii. Functional thinking: the ability to generalise relationships between co-varying quantities, and representing, justifying, and reasoning with these generalisations through natural language, variable notation, drawings, tables, and graphs.

(Blanton *et al.*, 2018)

Bush and Karp (2013) outline the areas of mathematics that are required for learning algebra and students' common misconceptions in these areas. The areas identified are ratio and proportional reasoning, fractions, decimals, percentages, integers, exponents, order of operations, properties of numbers, comparing and ordering numbers, equality, variables, expressions, equations, and functions. This review was based in the USA and as such was aligned with the syllabus for middle grade (ages 10-14) students. This syllabus is closely aligned with the Number, and Algebra and Functions strands studied by Irish students in junior cycle. However, one area from the junior cycle syllabus – 'patterns' – is omitted. Patterns are identified as an important area of mathematics to assist with students' understanding of generalising (Warren & Cooper, 2008; Welder, 2012) and are therefore included here as a prerequisite content area for learning algebra.

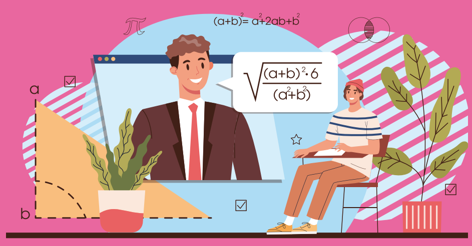
To define the construct of initial algebra for this study, the related content domains from the Irish syllabus are aligned to the prerequisite and algebra content areas identified in the literature, and with the conceptual model adopted. This is summarised in Table 1.

Table 1: Alignment of conceptual model for initial algebra with relevant content areas and the Irish subject specification for junior cycle

Component of Conceptual Model	Content Areas	Junior Cycle Mathematics Subject Specification
Generalised arithmetic	Ratios and proportional relationships	*Number Systems:* Consolidate their understanding of the relationship between ratio and proportion. Examining algebraic relationships: proportional relationships
Generalised arithmetic	Fractions	*Number Systems*: Investigate models to think about operation on fractions. Use the equivalence of fractions, decimals, and percentages to compare proportions.
Generalised arithmetic	Decimals and Percentages	*Number Systems*: Calculate percentages. Use the equivalence of fractions, decimals, and percentages to compare proportions.
Generalised arithmetic	Integers	*Number Systems*: Investigate models, such as the number line, to illustrate the operations on integers.
Generalised arithmetic	Exponents	*Indices:* Use and apply the rules of indices.
Generalised arithmetic	Order of operations	*Number Systems*: Appreciate the order of operations, including use of brackets.
Generalised arithmetic	Properties of numbers	*Number Systems*: Investigate the properties of arithmetic and the relationships between them.
Equivalence, expressions, equations, and inequalities	Compare and order numbers	*Number Systems*: Use the number line to order natural numbers, integers, and rational numbers. Use the equivalence of fractions, decimals, and percentages to compare proportions.
Equivalence, expressions, equations, and inequalities	Equality	*Number Systems:* Consolidate the idea that equality is a relationship in which two mathematical expressions hold the same value.

Component of Conceptual Model	Content Areas	Junior Cycle Mathematics Subject Specification
Equivalence, expressions, equations, and inequalities	Variables	*Expressions*: Using letters to represent quantities that are variable.
Equivalence, expressions, equations, and inequalities	Algebraic expressions	*Expressions*: Arithmetic operations on expressions. Transformational activities
Equivalence, expressions, equations, and inequalities	Algebraic equations	*Equations and inequalities*: Selecting and using suitable strategies for finding solutions to equations and inequalities.
Functional Thinking	Functions	*Representing situations with table diagrams and graphs:* Use tables, diagrams, and graphs as a tool for analysing relations – present and interpret solutions, explaining and justifying methods, inferences, and reasoning.
Functional Thinking	Patterns	*Generating arithmetic expressions from repeating patterns:* Use tables and diagrams to represent a repeating-pattern situation; generalise and explain patterns and relationships in words and numbers; write arithmetic expressions for terms in a sequence.

6
Challenges with Initial Algebra



Students being introduced to algebra find it difficult because it introduces "abstract representations and more complex mathematical relationships, but also because it can magnify misconceptions that have their roots in earlier instruction" (Booth *et al.*, 2015, p. 80). This section discusses the knowledge, skills, and abilities (KSAs) required for initial algebra. Throughout the discussion the different areas with which students struggle with initial algebra and how specific misconceptions arise in each area are outlined. Students' beliefs, understanding, meanings, explanations, and theories form the basis of student conceptions. When these conceptions conflict with accepted meanings in mathematics then a misconception has occurred (Osborne & Wittrock, 1983; Mulungye, 2016). Student misconceptions in mathematics can cause serious difficulties in learning algebra and are often persistent and resistant to conventional teaching methods as they produce systematic errors that can interfere with learning (Kuo *et al.*, 2017).

Much research has taken place to establish the common misconceptions students encounter when studying algebra (Booth, 1988; Perso, 1991; Russell *et al.*, 2009). Students will have acquired misconceptions through prior learning and from informal theories constructed from everyday experience. Other research has shown than when a student holds a particular misconception, they will often hold related misconceptions (Perso, 1991). A solid understanding of the number system is not the sole prerequisite for algebra although it is vital for understanding algebra. A "mastery of number operations, concepts, properties, and so forth greatly facilitates students' development in algebra" (Bush & Karp, 2013, p. 620). It is important to note the cognitive gap between arithmetic and algebra, as students must be introduced to concepts from arithmetic that will enable their algebra readiness (Herscovics & Linchevski, 1994). Careful attention must be paid to pre-concepts as "they are a relevant and necessary prerequisite for constructing formal concepts" (Linchevski, 1995, p. 114).

6.1 Ratios and Proportional Relationships

Hilton *et al.* (2013) succinctly defines proportional reasoning as "the use of ratios in the comparison of numbers" (p. 523), while Post *et al.*, (1988) states proportional reasoning "involves a sense of co-variation, multiple comparisons, and the ability to mentally store and process several pieces of information" (p. 79). Proportional reasoning is a key aspect of mathematics covered at junior cycle (NCCA, 2017). It is regarded as one of the major components of formal thought and once achieved this form of thinking can assist with learning throughout all disciplines of science, mathematics, and life (Bush & Karp, 2013; Post *et al.*, 1988). However, students do not always develop proportional reasoning skills naturally, and often mistakenly apply additive thinking to problems that require proportional reasoning (Hilton *et al.*, 2013). To understand proportional reasoning fully a student must be able to: reason multiplicatively, understand rational numbers, analyse functional relationships, equivalence, ratio and its parts, that is the different relationship's possible part/whole, part/part or even whole/part. It is not an easy concept for the adolescent mind to grasp as it is highly conceptual and a skill that develops gradually (Singh, 2000).

6.2 Fractions, Decimals and Percentages

"The proper study of fractions provides a ramp that leads students gently from whole number arithmetic up to algebra" (Wu, 2001, p. 10). In 2008, the National Mathematics Advisory Panel (NMAP) in the USA highlighted that fraction knowledge and ability was critical to the learning of algebra. Studies followed to provide the empirical evidence to support this statement and to identify which exact component of fraction knowledge best predicts ability in algebra (DeWolf *et al.*, 2015). Fractions are an integral part of algebra and can be found as coefficients, constants and solutions to equations, the slope of a line, and, in general, proportions are written in fraction form in algebra (Bush & Karp, 2013). In Ireland, lack of knowledge of fractions on entering post-primary school is reported as impacting negatively on algebra performance (Shiel & Kelleher, 2017).

A study by Booth *et al.* (2014) investigated what aspect of fractions exactly best predict algebra readiness and results suggest that it is knowledge of fraction magnitudes. This extended on findings from previous studies in two ways. First, it identified the role that fraction knowledge has in algebra readiness, that fraction magnitude is important as well as computational skills. Secondly, in exploring students' placement of disparate fractions on the number line a clearer picture of the knowledge components held by students emerges. Specifically, the placement of unit fractions, fractions with a numerator of one, was significantly correlated with a student's future knowledge, equation and word problem solving.

A further study by DeWolf *et al.* (2016) aimed to pinpoint what aspect of fraction knowledge was best at predicting algebra performance. They concluded that understanding of decimal magnitudes when assessed by a number line task and a relational understanding of fractions are strong predictors of algebra performance. The tasks for the relational understanding of fractions were a new addition to the study and contained tasks which tested fraction equivalence, division, inverse, multiplying by reciprocal and identifying ratios in countable sets, both part to part and part to whole. These results link the content areas of proportional reasoning, fractions, and decimals as important prerequisites for learning algebra.

Students can encounter difficulties in comparing and ordering when presented with numbers in different formats such as fractions, decimals, and percentages. If students do not understand the basis of the number itself, they are unlikely to be able to extend their understanding to which is greater than or less than or equivalent. The difficulty arises for many students in that they learn a particular procedure for dealing with a type of number, for example using the common denominator when adding fractions, therefore they rely on memory when trying to deal with these numbers (Bush & Karp, 2013). The lack of understanding leads to an inability to apply the rules correctly when working with variables in algebra.

6.3 Integers

A solid understanding of and procedural fluency with integers is a necessary requirement for success in algebra (Bottoms, 2003; Bush & Karp, 2013). To gain algebraic competence in terms of solving problems and equations students must understand the extension of the numerical domain from natural numbers to integers (Gallardo, 2002). When children are introduced to the concept of a negative number it can be difficult and teachers rely on using money and debt, temperature or sea level to explain the concept (Bush & Karp, 2013). Many students avoid working with the negative sign altogether with some students believing "that negative signs represent only the subtraction operation and do not modify terms" (Booth *et al.*, 2014 p. 10). These problems may be attributed to students' difficulty with the concept of a negative number itself, and students find it difficult to produce a negative solution as this abstract solution does not make sense (Vlassis, 2008).

6.4 Exponents

An understanding of exponents is required throughout algebra in both the transformational skills and the generational global/meta level skills (Bottoms, 2003; Bush & Karp, 2013). Transformational skills include factorising, multiplying, dividing, and simplifying algebraic expressions all require a knowledge of exponents. The generational skills where knowledge of the shape of functions is required also relies on this knowledge. The routinisation of procedures leads students to make errors with the basic rules of indices. For example, a student understands the division process with integers and may confuse this procedure when faced with for example, and divide the powers in error (Gray *et al.*, 2001)

6.5 Order of Operations

Knowing the order in which to carry out arithmetic operations is a fundamental skill required for algebra (Bottoms, 2003). There are many common misconceptions when dealing with the order of operations, the most common of which is carrying out operations in order from left to right. Some students believe that order of operations does not matter, that they will deduce the same answer regardless of the order in which they carry out the operations. Some believe the context of the problem determines the order of operations or in the absence of context, operations should be performed from left to right (Booth & Carlton Johnson, 1984). Another study has similar findings with students carrying out operations from left to right, however it found that the rule of "brackets first" is clear to many students (Linchevski & Livneh, 1999).

Linchevski (1995) suggests that more time should be given to students in allowing them to discover that performing operations in a different order results in different answers. Schwartzman (1996) suggests that the use of common mnemonics to remember the order of operations is not helpful. To start with there are several different ones in use, for example BOMDAS; brackets, orders, multiply, divide, add, subtract and BIRDMAS; brackets, indices, reciprocals, division, multiplication, addition, subtraction, to name just two. Schwartzman (1996) suggests that students should learn the hierarchy of operations more naturally, that is the more complicated operations of multiplication and division should be attended to before addition and subtraction.

6.6 Equality

Many studies focus on the concept of the equal sign in the early stages of learning algebra (Asquith *et al.*, 2007; Byrd *et al.*, 2015; Stacey & MacGregor, 1997; Stephens *et al.*, 2013). Students often misinterpret the meaning of the sign, taking it to mean 'perform a calculation' that is viewing it as an operational symbol (Stephens *et al.*, 2013). Those who interpret the equal sign correctly and see it as a relational symbol have more flexibility when working with equations. A student who has a structural understanding of algebra can 'see' abstract ideas hidden behind the symbols, therefore the symbols become transparent, and the transformations allowed to solve the equation become apparent (Kieran, 2007). "Students' interpretations of the meaning of the equal sign and their abilities to work with equations in a structural way are clearly related issues" (Stephens *et al.*, 2013, p. 174). A study by Booth *et al.* (2014) found that equality/inequality errors emerged as students become more experienced with solving equations and worsen as the school year continues and can affect a student's overall mathematics achievement.

6.7 Variables and Expressions

Kieran (1992) asserts that many misconceptions and common errors in algebra are generally rooted in the meaning of symbols or the letters used. Much research has been conducted into students' difficulties in working with algebraic variables and it is identified as a key area for student misconceptions (Asquith *et al.*, 2007; Booth *et al.*, 2014; Bush & Karp, 2013; Jupri *et al.*, 2014; Perso, 1991).

An essential element of algebraic thinking is a rich understanding of variables (Hunter, 2010). Common misconceptions with variables include viewing variables as labels, and the belief that a variable is just a missing value rather than varying values (Asquith *et al.*, 2007; Bush & Karp, 2013). Küchemann (1978) noted the different ways in which variables are used and established a hierarchy of variables as outlined in Table 2.

Table 2: Küchemann's (1978) hierarchy of variables

Name	Example
Letter EVALUATED	The variable can be directly evaluated, e.g., x + 7 = 11, x = ?
Letter IGNORED	The letter(s) do not have to be evaluated, e.g., a + b = 43, then a + b + 2 = 45, found by just adding 2 to 43, ignoring a + b.
Letter as OBJECT (LABEL)	Letters stand for the names of items, e.g., P = perimeter, a = apple.
Letter as SPECIFIC UNKNOWN	Variable stands for an unknown quantity and cannot be evaluated, e.g., Alice had x euro an Bob has 3 more, then Bob has x + 3 euro.
Letter as GENERALISED NUMBER	Letters can represent a set of numbers and not just a specific number, e.g., a + b = 10, a < b the a =?
Letter as VARIABLE	The relationship varies, e.g., which is larger, 2x or 2 + x, it depends on what x equals.

Stacey and MacGregor (1997) identified teaching methods as another possible source of confusion amongst students in understanding variables. For example, a teacher who frequently uses the first letter of a word as the unknown can result in students understanding a variable as a label rather than an unknown quantity. Another issue identified in their research was that students did not understand that they could use variables to help solve problems, an essential skill for using algebra effectively. Using algebraic notation frequently, reviewing computational skills and emphasising that variables, that is letters in algebraic expressions stand for unknown quantities rather than labels, better prepared students for algebra (Bush & Karp, 2013).

Often students' struggles with algebraic expression result from misconceptions with variables (Bush & Karp, 2013; Stacey & MacGregor, 1997). When working with expressions many students fail to simplify properly in that they do not add and subtract like terms, for example a common error is to simplify 2x + 4 to 6x (Kieran, 1992). Another error is to detach terms (which may or may not contain a variable) from the operations (Jupri *et al.*, 2014; Linchevski, 1995). Furthermore, believing that the sign 'belongs' to the preceding term rather than the term after leads to incorrectly simplifying expressions such as 2 - 3x + 2x to 2 - 5x.

When a student is at the stage of understanding a variable as a label only, this results in further errors with algebraic expressions (Küchemann, 1978). For example, students posed with the following problem "Apples costs *a* cents each and bananas cost *b* cents each. What does the expression 4a + 3b stand for?" Many students interpret the expression to mean four apples and three bananas (Bush & Karp, 2013). All these misconceptions lead to further difficulties in accepting an algebraic expression as an answer to a problem. This is referred to as "lack of closure" where a student proceeds to solve an expression due to a focus on the solution (Booth, 1988). In short students' misconceptions in almost all the areas discussed so far hinder their ability to work effectively with expressions.

6.8 Equations

The underlying misconceptions and difficulties students hold in relation to variables, expressions and indeed all the prerequisite content areas lead to difficulties in solving algebraic equations. The ability to solve equations is reliant on both procedural and conceptual understanding (Booth & Davenport, 2013). A solid understanding of how to use variables to write algebraic expressions, form subsequent equations and solve when necessary is the essence of success in algebra at junior cycle level.

Students are often introduced to the notion of solving an equation through opportunities to experience the substitution of numbers for letters for example 2 + x = 6. Much of the research into difficulties with solving equations suggests that it becomes much more difficult when students encounter one variable equations with the variable on both sides of the equation as students can no longer rely on arithmetic to solve the equation (Herscovics & Linchevski, 1994; Kieran, 1980). Over fifty concepts related to solving equations have been identified (Chung & Delacruz, 2014). Without the underlying conceptual understanding students learn a series of memorised transformational rules for solving equations which can lead to misconceptions. Students can lose the concept that inverse operations are being performed without changing the equality of the equation (Capraro & Joffrion, 2006). Further problems arise with the computations involved especially if negative numbers, fractions, or decimals are involved (Wu, 2001).

Chung and Delacruz (2014) outline three levels of cognitive development in students who can solve equations: *adaptability, adaptive expertise, and metacognition.* A student competent at solving equations can cope with equations in different forms, ones that involve multiple terms and are comfortable with the transformative rules required to simplify the equation and with the different forms the equation takes as it moves towards the solution. These students have *adaptability* when solving equations, and they may move on from this level of development to *adaptive expertise*, whereby they develop a deeper understanding of the underlying structure of the equation or problem and know why or why not to apply particular transformative rules to solve the problem. The final level of development is *metacognition*, these students are at a level where they can plan and monitor each step of the solution process. It has been suggested that the conceptual approach to teaching algebraic equations, using tables and graphs, would foster better student understanding and therefore better equation solving skills (Carraher *et al.*, 2006; Kalchman & Koedinger, 2005).

6.9 Functions and Patterns

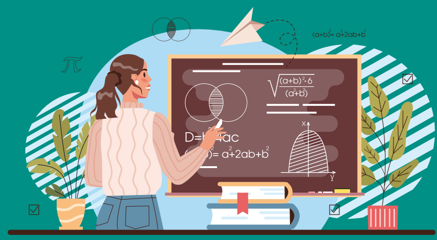
The concept of a function is one of the most important in mathematics (Ellis, 2011; Peled & Carraher, 2008), "it permeates all of mathematics, from first year algebra through calculus and beyond, as well as most applications of mathematics" (Willoughby, 1997, p. 215). A function is defined as a correspondence between two sets (Kieran, 1992), and there are two general approaches to teaching and learning functional relationships mentioned in the literature; a correspondence approach and a covariation approach (Ayalon *et al.*, 2015). The correspondence approach deals with an input-output model, whereby an output value y is calculated for a given input value x, often listed in a table of values or as couples (x,y). A covariation approach involves understanding that as the input (x-value) changes then so does the output (y-value), it entails analysing, manipulating, and understanding the relationships between changing quantities.

Ronda (2009) defines four growth points for students when working with functions: (i) equations are procedures for generating values, (ii) equations are representations of relationships, (iii) equations describe properties of relationships and (iv) functions are objects that can be manipulated and transformed. Van de Walle *et al.* (2013) includes the pattern itself as another representation of a function, referred to as the context. The ability to plot graphs from tables and patterns is required for procedural fluency, which some students experience difficulty with (Bottoms, 2003), while others have difficulty interpreting graphs (Kilpatrick *et al.*, 2001). These difficulties, amongst others, prevent students from translating from one form of representation to another and hinder conceptual understanding of functions in the process.

Patterns are a popular pedagogic tool for teaching children to generalise and are one of the routes to aid understanding of variables and algebra overall (Mason, 1996). The ability to see patterns, explain them and represent them with algebraic symbolism is important for mathematical problem solving and it "has pedagogic importance towards understanding functions but is also an end in itself" (Ayalon *et al.*, 2015, p. 324). When working with patterns, the relationship which lies between the pattern and its position in a functional relationship must be identified, and accordingly an expression or formula must be created using variables. In doing this a context for the use of variables is set for students, assisting their understanding of a variable as a varying quantity rather than a specific unknown (TaniĐli & Köse, 2011).

This section discussed the issues with the teaching and learning of initial algebra and the prerequisite and algebra content areas where students must develop their knowledge and skills to succeed with the subject. Failure to recognise students' algebraic misconceptions (Asquith *et al.*, 2007), hinders students' development with thinking algebraically and progression with the subject as a whole (Ralston *et al.*, 2018). This research investigated students' knowledge of initial algebra and aimed to identify common misconceptions with the aim of providing information for educators to help inform instruction. The next section outlines the methods employed in collecting and measuring this knowledge.

7 Methodology



7.1 Introduction

This research set out to profile Irish students' knowledge of algebra early in their post-primary education with an aim of identifying students' strengths and weaknesses in relation to initial algebra. To collect this evidence a suitable instrument for measuring a student's knowledge of algebra was required. An assessment in the form of a standardised criterion referenced screener for algebra has been developed as part of this research. The requirement for a *standardised* assessment arises to profile the students' knowledge of algebra effectively. When an assessment is standardised, it means that all students are administered the same items in the same way and that the resulting scores can be objective and consistent (Popham, 1999). This ensures that a dependable profile of the students' knowledge is an outcome of the administration of the screener (OECD, n.d.).

7.2 Development of the screener

Screeners are typified as formative assessments tools, that are brief and technically adequate and are used to identify whether students are on track with curricular expectations or not (Glover & Albers, 2007; Ketterlin-Geller *et al.*, 2019). Figure 1 outlines the steps taken to develop the standardised screener for algebra used in this study.

Figure 1: Overview of the Development of the Screener for Algebra

1. Defining objectives: Identifcation of the knowledge, skills, and abilities in the pertinent content areas required for success in algebra.
2. Preparation and selection of material: Development of an Item Bank of task items from the literature covering each content area.
3. Experimental organistion & Item Development: Selection of task items from Item Bank for inclusion in Draft 1 and submission to expert panel for review. Development of Draft 2 based on review.
4. Pretest: Phase 1 - Administration of Draft 2 in a pilot school, and subsequent analysis of results. Development of Draft 3 based on pilot.
5. Assembling of the asessment: Administration of Draft 3 in 19 post-primary schools.
6. Determining the scoring system.
7. Determining validity and reliability.
8. Detecting and removing unfair items: Analysis of results.
9. Final organisation of screener.

Cronbach and Meehl (1955) state that the construct to be measured consists of a universe of content, from which items that are domain relevant should be sampled, and from these a test that is representative of the domain can be developed. By defining the construct of interest "initial algebra" and using a conceptual model aligned with relevant content areas and the Irish syllabus (see Table 1) a systematic search of the literature for studies that assessed students in each area was conducted. Task items contained in all the relevant studies were recorded in an item bank for possible inclusion in the assessment. This item bank together with a proposed first draft was then forwarded to an expert panel for review. The revised draft was returned containing 21 task items assessing the pertinent content areas and utilised for piloting and further development of the screener. Two types of item format are utilised on the screener; multiple-choice known as selected-response (SR), and constructed-response, objective scoring (CROS) (Haladyna & Rodriguez, 2013). Of the SR items, some used the Complex Multiple-Choice format where these items allow for answers that are incorrect, part correct, or fully correct (polytomous item). The remaining items used the Conventional Multiple-Choice format with four or five possible answers and only one of these is the absolute correct answer (dichotomous item). Part of the development of the screener was developing an SR format for existing constructed-response items identified in the literature.

An aim of this study is to validate the items for use with Irish second-year students specifically, to ensure that a given construct has been operationalised fairly for the population of interest (Cohen *et al.*, 2018). It is important to note that subject matter expertise outweighs statistical guidelines when deciding if items should be retained, if it is believed they are measuring something important (Haladyna & Rodriguez, 2013). The revised draft of the screener returned from the expert panel was administered using pen and paper in a pilot school (n = 67) with a short survey. Task items were revised based on the results of this administration and feedback from teachers and students in the pilot school. In addition, three different versions of the screener were produced for use in the main study. A major issue arose during the pilot administration known as "speededness". Students, when completing items on the latter half of an assessment, will tend to speed up and possibly start guessing in order to finish on time, while others may not get to the items at the end at all (Bolt *et al.*, 2002). Another issue to consider when administering such assessments in the classroom is cheating/copying. Therefore, it was decided to produce three versions of the screener. Each version contained the same items arranged differently. The first item on the screener assessing decimal number magnitude was contained a different decimal number on each version of the screener and was the only item to differ. In producing three versions of the screener the following had to be addressed: (i) item difficulty rating (facility index), (ii) item grouping, and (iii) anchor items. Advantage is conferred to students taking an assessment where items are ordered from easy to difficult (Monk & Stallings, 1970). Therefore, the FI calculated for each task item in the pilot was used to rank the items. Item grouping is where items based on similar content areas, for example fractions, must be grouped together within the assessment, but may be rearranged within the subgroup (Briggs *et al.*, 2006). Finally, the use of anchor items was employed. Item 1 on number line placement, 6 on proportional reasoning, 11 on comparing numbers and 15 on equation solving/integers were held in position for all three versions of the screener (Monk & Stallings, 1970). The anchor items are highlighted in grey rows in Table 3.

Table 3: Order of Items in Versions 1, 2 and 3 of the screener

Version 1 Item Order	Version 2 Item Order	Version 3 Item Order
1. Decimal 0.386 placement on number line	1. Decimal 0.1529 placement on number line	1. Decimal 0.78 placement on number line
2. Fraction knowledge – way to find half a number	2.1 Patterns – completing the table 2.2 Patterns translating from table to situation 2.3 Patterns translating from situation/ table to equations (algebra)	2. Equation solving – end of solution path
3. Fraction knowledge – multiplying fractions with a variable	3. Properties of numbers	3. Equation solving – start of solution path
4. Equivalent fractions	4. Order of operations	4. Forming equations
5. Fraction and variable knowledge – 1/n	5. Variable and expressions	5.1 Patterns – completing the table 5.2 Patterns translating from table to situation 5.3 Patterns translating from situation/ table to equations (algebra)
6. Proportional reasoning	6. Proportional reasoning	6. Proportional reasoning
7. Exponents – power raised to a power	7. Equality – is left equal to the right	7. Equivalent fractions
8. Exponents – squaring an expression	8. Equality – complete the equation	8. Fraction knowledge – multiplying fractions with a variable

Version 1 Item Order	Version 2 Item Order	Version 3 Item Order
9. Order of operations	9. Exponents – squaring an expression	9. Fraction and variable knowledge – 1/n
10. Properties of numbers	10. Exponents – power raised to a power	10. Fraction knowledge – way to find half a number
11. Comparing and ordering numbers	11. Comparing and ordering numbers	11. Comparing and ordering numbers
12. Equality – complete the equation	12. Forming equations	12.1 Forming algebraic expressions 12.2 Forming algebraic expressions 12.3 Forming algebraic expressions 12.4 Forming algebraic expressions
13. Equality – is left equal to the right	13. Equation solving – end of solution path	13. Variable and expressions
14. Variable and expressions	14. Equation solving – start of solution path	14. Simplifying algebraic expressions
15. Integers and equation solving	15. Integers and equation solving	15. Integers and equation solving
16.1 Forming algebraic expressions 16.2 Forming algebraic expressions 16.3 Forming algebraic expressions 16.4 Forming algebraic expressions	16. Fraction and variable knowledge – 1/n	16. Order of operations
17. Simplifying algebraic expressions	17. Fraction knowledge – way to find half a number	17. Properties of numbers

Version 1 Item Order	Version 2 Item Order	Version 3 Item Order
18. Equation solving – start of solution path	18. Fraction knowledge – multiplying fractions with a variable	18. Equality – is left equal to the right
19. Equation solving – end of solution path	19. Equivalent fractions	19. Equality – complete the equation
20. Forming equations	20. Simplifying algebraic expressions	20. Exponents – power raised to a power
21.1 Patterns – completing the table 21.2 Patterns translating from table to situation 21.3 Patterns translating from situation/ table to equations (algebra)	21.1 Forming algebraic expressions 21.2 Forming algebraic expressions 21.3 Forming algebraic expressions 21.4 Forming algebraic expressions	21. Exponents – squaring an expression

The revised screener (Draft 3) consisted of seventeen SR items and four CROS items. Table 3 summarises each item and its content. The screener was administered using pen and paper to 576 second-year students in 19 post-primary schools (29 classes) across Ireland in October 2016 and again in April 2017. There were two reasons for administering the screener twice to the same group of students; first was to enable the test-retest reliability measure of stability for the screener results (Cohen *et al.*, 2018), and second was to measure the change in the students' performance over the academic year.

Finally, reporting of results must be an integrated part of the assessment design and development (Lane *et al.*, 2016). Therefore, an objective scoring system was developed for the screener. This consisted of two types of scores; first a cognitive score which awards a score when an item is correctly answered, and second, an error score which identifies a type of error associated with an item. A cognitive score (CS) is one which identifies a cognitive process, for example, knowledge, recall, interpretation, or synthesis (Anderson & Morgan, 2008). A CS was applied consistently to all items whereby a completely correct response was awarded 2, partly correct 1 and an incorrect response 0. Error scores have been developed for items to identify both potential procedural errors and misconceptions. An error score of '0' means that the student demonstrated evidence of understanding the concept and an error score of '1' reflects incomplete knowledge, skills, or abilities.

It is important to note that the screener administered in this study was still under development and that some of the items were revised after the analysis of the results. Classical Test Theory (CTT) is used to analyse the results of the task items in terms of item difficulty, discrimination, and functioning distractors. Important construct validity evidence arises from the statistical analyses of the item responses. The results of these analyses inform the presentation and layout of the task items for the final draft of the screener to ensure the construct of initial algebra is operationalised fairly. The impact of these analyses is discussed in the findings section where the layout of some of the items may have impacted results.

7.3 Sampling strategy and procedure

The initial step in selecting a sample is to identify the population of interest, which for this study is second year post-primary students studying mathematics in Ireland. Mathematics is offered at higher, ordinary and foundation levels and students studying at all levels comprise the population for this study. An overall objective was to assess circa 400 second year post-primary students. The figures from the Department of Education and Skills (DES) showed a total of 57,212 students registered as attending second year in post-primary school in Ireland for the academic year 2015/2016. The post-primary school system in Ireland is a natural clustering of second year post-primary students, this follows from the sampling methodology used by the ESRI research group which used the national education system as their point of entry to their required cohort of students (Murray *et al.*, 2010). In September 2015 there were 735 post-primary schools listed in Ireland per the information available from the DES website and this list provided the primary sampling unit (sampling frame) for this study (Department of Education and Skills, 2016). 12.9% of these schools were community schools, 36.1% were vocational schools and the remaining 51% were secondary schools.

A two-stage cluster design is the most commonly employed sample design in educational studies as it is practical and allows for analysis at more than one level of data aggregation, that is between student or between classes or indeed both if required (multi-level analysis) (Ross, 2005). This type of design involves the selection of schools (first stage) for inclusion in the study, followed by the selection of a cluster (class) of students within the school (second stage). The intraclass correlation, rho, gives a measure of the degree of homogeneity of variance within classes (clusters). Within school/class variability of student characteristics will be less than between school/class variability. No two schools are identical, as schools differ for example, on the sociodemographic composition of the student bodies, student achievement, learning related motivation and affect, etc. (Brunner *et al.*, 2018). Mathematics classes in post-primary schools in Ireland are known to have low intraclass correlation coefficients, with results from PISA 2015 reporting rho = 0.15 (OECD, 2017). Therefore, rho values of 0.1 and 0.2 were considered as acceptable from the tables provided in Ross (2005) in selecting the sample size and therefore and ideal sample would have been 24 classes with circa 480 students. A non-probability sampling technique known as snowball sampling where one participant (teacher/school principal) recommends at least one other participant who in turn recommends another was employed in this study (Creswell, 2012). This resulted in 19 schools and 29 teachers/classes being recruited, which resulted in a sample of 667 students.

A summary of the number of participants by class for each round of administration is detailed in Table 4 below.

Table 4: Breakdown of number of students per class per school in the study

School	PHASE 2 OCOBER 2016					PHASE 2 APRIL 2017				
	Class 1	Class 2	Class 3	Class 4	Total	Class 1	Class 2	Class 3	Class 4	Total
A	27	0	0	0	27	24	0	0	0	24
B	17	0	0	0	17	12	0	0	0	12
C	32	12	0	0	44	29	9	0	0	38
D	16	0	0	0	16	12	0	0	0	12
E	28	0	0	0	28	23	0	0	0	23
F	25	0	0	0	25	20	0	0	0	20
G	29	0	0	0	29	24	0	0	0	24
H	22	7	0	0	29	19	6	0	0	25
I	19	18	8	23	68	17	16	7	22	62
J	22	0	0	0	22	22	0	0	0	22
K	17	0	0	0	17	14	0	0	0	14
L	23	0	0	0	23	20	0	0	0	20
M	16	22	22	4	64	13	20	20	4	57
N	28	0	0	0	28	25	0	0	0	25
O	23	0	0	0	23	19	0	0	0	19
P	16	0	0	0	16	13	0	0	0	13
Q	23	0	0	0	23	22	0	0	0	22
R	23	0	0	0	23	17	0	0	0	17
S	9	14	10	0	33	7	13	8	0	28
Total number assessed in October 2016					**555**	**Total number assessed in April 2017**				**477**

7.4 Ethical considerations

The research was undertaken during the school year 2016-2017. Each school was visited in September 2016 at which stage the research was explained, administration of the screener outlined, and consent/assent packs were distributed to all participants. Full Garda Vetting was obtained prior to visiting all schools. Participants, both students and teachers were recruited through the schools. Full information on the research study and background documentation was prepared and sent via email to the Principal of each potential school. During the school visit each teacher was given a consent form to complete and sign. This form outlined the voluntary nature of their participation, that withdrawal from the study at any stage was possible without any negative implications and that their participation would be treated with strict confidentiality. The teachers then distributed the consent/assent packs in individual sealed envelopes to the students, with additional packs left for any absentees. The students were instructed to return the consent/assent packs to the school office in the individual sealed envelopes. After one week the school secretary posted all returned consent/assent packs to me via An Post. Students were assessed in early October 2016 at the start of the academic year and again in April 2017 towards the end of the academic year.

A unique identification number (UIN) was assigned to each student in the study at the end of each school visit. A copy was held by the teacher for administering the screener and a copy was held by me to record consent/assent. Assigning each school in the study a unique letter and each student a UIN protected the privacy of the schools and students involved in the research. The screeners were delivered via An Post (posted on the same date) and administered the within one week of receipt. In accordance with the ethical approval for this research all screeners were delivered in a 'pack' which contained (i) a short teacher questionnaire (ii) a stamped addressed envelope for return of screeners, (iii) first year mathematics test results form (showing UIN only) and (iv) the screeners in individual unsealed envelopes with the students' UIN pre-printed on the screener and the envelope. The students themselves removed the screener from the envelope, completed it, and then placed it back into the envelope and sealed it before returning to the teacher. The teachers completed their questionnaire during or shortly after the administration of the screener. The teacher then returned items (i), (iii) (iv) and in the stamped addressed envelope provided (ii). The return of the screeners was prompt. The same process was then adhered to for the second administration in April.

In administering the screeners every second-year student present in the class on the day completed it in accordance with ethical approval so that no student was treated differently. The consent/assent or lack thereof was recorded in an excel spreadsheet under the students' UIN. All screeners were returned to me and where there was no consent/assent provided, the screener in its envelope was removed and destroyed. The results were then inputted into SPSS and the database created in conjunction with the class level information obtained from the teacher survey and school level information. For the second round of administration the same method was adhered to. Where a student had joined the class during the academic year, the teacher provided notice to me. It was decided that for these students all teachers in the study would receive three additional screeners in their second pack with new UINs assigned. All screeners would be returned but these additional screeners were destroyed as no consent/assent forms had been distributed for these students. There was also the issue that some students were absent for the second round of testing but had participated in the first round and vice versa.

7.5 Profile of participants

All schools in the sample were English medium schools and 3 (15.8%) carried a DEIS indicator. Despite the method of recruitment, the resulting sample of schools was very close to the primary sampling unit as can be seen in Table 5.

Table 5: Population of schools (2015-2016) versus sample schools

School Type	Total Population		Sample	
	n	%	n	%
Single-sex Girls Secondary	135	18.37	4	21.05
Single-sex Boys Secondary	103	14.01	3	15.79
Co-Ed Secondary	137	18.64	4	21.05
Subtotal	*375*	*51.02*	*11*	*57.89*
Single-sex Girls Vocational	2	0.27	0	0
Single-sex Boys Vocational	0	0.00	0	0
Co-Ed Vocational	263	35.78	7	36.84
Subtotal	*265*	*36.05*	*7*	*36.84*
Single-sex Girls Community/Comprehensive	1	0.14	0	0
Single-sex Boys Community/Comprehensive	1	0.14	0	0
Co-Ed Community/Comprehensive	93	12.65	1	5.26
Subtotal	*95*	*12.93*	*1*	*5.26*
Total	**735**	**100**	**19**	**100**

In terms of the distribution of the numbers of students in the sample versus school type it is helpful to compare with the distribution of the total number of students in the population frame. The distribution of students enrolled in second year post-primary was not available for the population, however the total numbers of students enrolled in the different school types was. Table 6 shows the breakdown of the sample by school type compared to the breakdown of all students enrolled in post-primary education by school type for the academic year

2015/2016. It can be assumed that the distribution of students enrolled in second year post-primary by school type will be very similar to that of all students in all academic years. The sample has an over representation of students from secondary schools: 66.1% of students in the sample are enrolled in secondary schools as opposed to 51.2% of students in the population. Accordingly, there is under representation of students enrolled in vocational schools in the sample, 29.7% as opposed to 33.3%. There is a greater under representation of students in community/comprehensive schools in the sample with only 4.1% of the sample drawn from this type of school. Traditionally, secondary schools were established to provide a more academic education, and vocational schools provided a technical education to develop manual skills. This distinction no longer exists with both types of schools providing a wide range of subjects of both an academic and technical nature (Citizens Information Board, 2015). Therefore, the underrepresentation of students in certain school types will not overly affect the generalisability of the results.

Table 6: Distribution of all students enrolled in post-primary education by school type compared to students in the sample.

School type	Total Number of students	% of students in all years	Number of students October 2016	% of students October 2016
Single-sex Girls Secondary	72,079	19.1	102	18.4
Single-sex Boys Secondary	56,094	14.8	130	23.4
Co-Ed Secondary	65,186	17.2	135	24.3
Subtotal	*193,359*	*51.2*	*367*	*66.1*
Single-sex Girls Vocational	541	0.1	0	0
Single-sex Boys Vocational	0	0	0	0
Co-Ed Vocational	125,194	33.1	165	29.7
Subtotal	*125,735*	*33.3*	*165*	*29.7*
Single-sex Girls Community/ Comprehensive	773	0.2	0	0
Single-sex Boys Community/ Comprehensive	595	0.2	0	0

School type	Total Number of students	% of students in all years	Number of students October 2016	% of students October 2016
Co-Ed Community/Comprehensive	57,541	15.2	23	4.1
Subtotal	*58,909*	*15.6*	*23*	*4.1*
Total	**378,003**	**100**	**555**	**100**

In terms of the sample of students involved 555 students' data were included in October 2016 and 476 in April 2017. The difference in the number of students was due to lack of consent/assent, and absences in schools on the day of administration of the screeners. Of the 555 students assessed in October 305 (55.0%) were male, 248 (44.7%) were female and 2 (0.4%) preferred not to say. In April 264 (55.5%) were male and 212 (44.5%) were female. The gender breakdown of students in Irish post-primary schools in the academic year 2016-2017 was 47.7% male and 52.3% female (Central Statistics Office, 2021).

7.6 Data Analysis Strategies

Quantitative data analysis was required for both the pilot and main study where the results from the screener were analysed using classical test theory. Additionally, multi-level models were employed with main study data to establish the validity and reliability of the screener. The quantitative data from the pilot school were input into Microsoft Excel and the results analysed. Specifically, from Classical Test Theory, the facility index (FI), for each question was calculated using the formula, where C is the number of students who answered an item correctly and N is the total number of students in the sample.

The quantitative data from the main study were input into SPSS and the objective scoring system as described in section 7.2 was developed. These quantitative data include each students' response to each item on the screener together with their basic demographic information and school and class information. The purpose of the first set of analyses was to establish the validity and reliability of the screener. There are three stages to this analysis (i) the statistical analysis of item responses, (ii) criterion and construct validity analysis and (iii) reliability analysis (Haladyna & Rodriguez, 2013). The statistical analysis of item responses uses Classical Test Theory. First the FI, as described above, was calculated for each item to assess item difficulty followed by item discrimination analysis and lastly each multiple-choice item was assessed for non-functioning distractors (Finch, 2016). Subsequently, the criterion validity of the screener was assessed using correlational analysis (Foster, 2017; Meinck & Rodriguez, 2013). Thereafter, the reliability of the screener was assessed using correlational analysis between the results of the screener at each administration (Creswell, 2012).

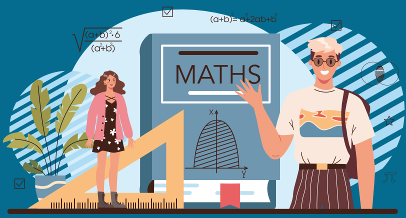
The purpose of the second set of analyses is to establish a profile of the students' knowledge of initial algebra. First the change in their performance between administrations is investigated using multi-level models (Field, 2009; Peugh, 2010). Subsequently, the descriptive statistics for each item are presented and ordered by the proportion answering each item fully or partly correct from highest to lowest. Thereafter, the error scores developed for the screener were analysed to investigate which type of errors and misconceptions are most prevalent among the sample. Again, simple descriptive statistics such as the frequency of each error and the proportion of each performance level making each type of error were investigated.

7.7 Study Limitations

There are some limitations of the study due the resulting sample design. These include:

- The issue of sampling only one class per school was raised by the teachers who wished to be involved in some schools. They felt that no class or student should be excluded from the study. This resulted in five of the nineteen schools testing more than one class. This can reduce the independence of the unit of study i.e., students. However, the influences of the school/teacher is considered in the analysis of the data.
- There was variability in the class (cluster) sizes. The maximum number of students in any one class for the study was 32, the mean number of students was 19, the median 22 and mode 23. Once again, this is considered during the analysis.
- Ideally, more classes (n = 24) should have been included in the sample as discussed under sample size. This could have an effect on the generalisability of the results.

8 Findings



The data collected through the administrations of the screener result in findings that portray broad trends rather than deep explanation of students' knowledge of initial algebra. The findings presented here focus on the attainment over the academic year, the content areas that our students perform well on, followed by identifying the content areas with which they struggle.

8.1 Attainment during the academic year

Many factors affect student mathematical attainment over the school year alongside instruction including personal variables, such as age, self-concept and environmental variables relating to home and school (Wilkins & Ma, 2002). The total CS on the screener in October versus April can give a measure of attainment in initial algebra over that period. A paired-samples t-test showed that there was a statistically significant increase in total CS from October ($x = 25.06$, $s = 10.34$) to April ($x = 28.94$, $s = 11.37$), $t(476) = -10.87$, $p < 0.005$ (two tailed). The mean increase in total CS was 3.88 with a 95% confidence interval ranging from 3.18 to 4.58. The eta squared statistic (.02) indicated a small effect size (Pallant, 2007). It can be concluded that there is a gain in students KSAs with initial algebra over the period of October 2016 to April 2017. The gain is small but meaningful given that internationally, not all students at this age have improved performance in mathematics over the academic year (Wilkins & Ma, 2002).

Multilevel models were used to investigate if gender or age impacted the total CS in April. There was a 0.76 increase in the total CS in April for girls, but this is not significant at the 5% level. It can be concluded that there is no difference in the results of the screener based on gender. This is in line with many other studies showing the gender difference in mathematics performance has decreased (Erturan & Jansen, 2015). Similarly, age in months was investigated to see if this factor impacted total CS in April and it can be concluded that there is no significant difference in the results of students based on their age. Additionally, the predictor variable of a DEIS indicator was used in the model and once again no significant difference was found in the total CS between students in DEIS and non-DEIS schools. To conclude there is a significant increase in the total CS from October to April. Many factors can affect this attainment alongside school instruction but for this study, age in months, gender and DEIS status did not impact significantly on attainment.

8.2 Students' strengths in relation to initial algebra

To provide a profile of second year students' performance of initial algebra this section looks in more detail at their responses to the items on the screener. The items are ranked from highest to lowest based on the percentage correct response, to identify which tasks and content areas are best and least understood by the sample, see Table 7. Items that were answered either correctly or part correctly by 50% or more of the sample in the October administration are discussed first. The content covered by these items is understood by at least 1 in 2 of the sample and is therefore considered to be well understood. The items are ordered from highest percentage correct response in October.

Table 7: Items on the screener answered correctly by more than half the sample ranked by proportion answered correctly in October

Item Content	Correct or part correct in October	Correct or part correct in April	Not Answered in October	Not Answered in April
Equality – asked to circle true or false for a number sentence.	84.1%	90.6%	10.8%	5.5%
Equality – asked to explain their answer to above.	79.1%	88.5%	20.9%	11.5%
Proportional Reasoning – asked if an image of 3 glasses with different parts water and sugar were equal concentration.	72.3%	72.3%	2.7%	0.8%
Patterns – asked to complete a table for the perimeter of a pattern shown (first missing entry in table).	71.9%	78.6%	14.1%	5.0%
Expressions – asked to write the expression for a perimeter of an equilateral triangle with side *c*.	71.0%	80.1%	18.9%	13.2%
Comparing and ordering numbers.	64.7%	74.8%	9.9%	5.9%
Distributive property	64.1%	70.4%	14.8%	8.0%

Item Content	Correct or part correct in October	Correct or part correct in April	Not Answered in October	Not Answered in April
Patterns – asked to complete a table for the perimeter of a pattern shown (second missing entry in table).	62.9%	73.4%	14.8%	6.7%
Equality – asked to complete a number sentence (fill in the missing number).	56.8%	71.7%	21.3%	13.0%
Fractions – asked to identify ways to find half a number *n.*	55.7%	65.8%	12.8%	7.1%
Expressions – asked to write the expression for a perimeter of a pentagon with 4 equal sides of *h* and one side *t.*	50.3%	59.5%	22.2%	14.0%

Content areas that are well understood based on the above results are equality, proportional reasoning, comparing and ordering numbers, the distributive property when presented in a straightforward manner, for example 2(x - 3), patterns, and simple algebraic expressions. Although one item on fractions is answered correctly by more than half of the sample it is not discussed in this section as there were four items in total assessing fractions which were not well answered. Throughout the discussion presented here, the changes in the proportion answering an item correctly between the October and April administrations are identified as significant or not at the 5% level (using McNemar's test).

8.2.1 Equality

The item assessing equality where students are asked if a number sentence is true or false, is the best answered item at both administrations. The second item on equality where students must complete a number sentence is answered correctly by 56.8% of the sample in October but this rises to 71.7% at the April administration, which is significant ($p = 0.001$). The concept of equality and the meaning of the '=' sign is firmly embedded in the primary school curriculum in Ireland, which has been in place since 1999, where children from junior infants upwards are encouraged to explore equality using number balance (NCCA, 1999). This study provides evidence that the concept of equality is well understood by most Irish students aged approximately 14 years.

8.2.2 Proportional Reasoning

Lesh *et al.*, (1988) identified seven types of problems which require proportional reasoning: missing value, comparison, transformation, mean value, conversions from ratios to rates and fractions, conversions involving units and between-mode translation problems. The different types of problems require different forms of reasoning and as such various types of difficulties arise for students within the different types of problems. The item on the screener assessing proportional reasoning was a comparison type problem. The item presents students with the image of three cups with three different amounts of water and lumps of sugar. Students are asked to circle true or false for the following statement "when the lumps of sugar are stirred in, the liquid in cup B will be sweetest". Students were then asked to explain their reasoning. 72.3% of students answered the first part of the item correctly at both administrations. However, the second part of the item where students had to provide an explanation was not well answered. In October 49.2% provided a correct or partly correct explanation which fell to 34.4% of students at the April administration.

The evidence provided here shows that Irish second year students perform well on a comparative task, but most cannot explain their reasoning.

8.2.3 Patterns

The item on patterns presented students with an image of stacked hexagons and they were asked to complete a table for the number of sides in the perimeter of each stack of hexagons which is a linear pattern. The first blank in the table for the length of the perimeter of 2 hexagons (image shown), was very well answered with 71.9% rising to 78.6% (not significant) completing the first blank correctly in October and April respectively. The second blank for the perimeter of 5 hexagons (image not shown) was not as well answered in October with 62.9% getting it correct but this increases to 72.3% in April which is significant ($p = 0.02$).

The second part of the item on patterns asked students to identify the statement that best described the pattern for the increasing perimeter in the stack of hexagons. This was answered correctly by just less than half of students in October (48.5%) but this rose to 60.6% of students answering correctly in April (see Table 8). This change is significant ($p < 0.001$). The specification document states that the Algebra and Functions strand "focuses on representing and analysing patterns and relationships found in numbers" and these results suggest that this is being implemented successfully (NCCA, 2017, p. 11).

However, the third part of the task asked students to identify the correct formula for the pattern. It is known that students struggle with forming the correct general formula for a pattern and often persist with an incorrect formula even when checking their solutions (Moss & Beatty, 2006; Stacey, 1989). Students were not required to create a formula but rather select the correct formula from a list, with incorrect answers based on common errors. Approximately half of the students identified the incorrect formula at each administration (52.4% and 48.4% in October and April) which is disappointing given the focus on studying patterns at junior cycle (NCCA, 2017). Recent studies show that a student's ability to use all the mentioned algebraic representations and to move between different representations fluently displays a true understanding of mathematical functions (Nitsch *et al.*, 2015). Given the functions-based approach to teaching algebra in Ireland where students are encouraged to explore patterns, it is disappointing that this final aspect of understanding is lacking.

The use of patterns to introduce students to variables and expressions and further their understanding has yet to reach its full potential, and this is discussed further in section 8.3.4. However, it is noted that the results presented here are based on assessing students early in the context of curriculum change.

8.2.4 Expressions

One of the items assessing expressions was taken from the thirty-year research project in the UK, Increasing Students' Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS). It is a constructed response item and uses an illustration. Four shapes are presented with their sides labelled and the student is asked to form an expression to represent the perimeter of each shape with each shape progressively more difficult for the student (Hodgen *et al.*, 2009). This first part of this item presented students with an image of an equilateral triangle of side *c* and asked them to write an expression for the perimeter in simplest form i.e., *3c*. It was answered well at both administrations with 71% and 80.5% (increase not significant) answering it correctly in October and April respectively, compared to 94% of UK students of a similar age in the 1978 ICAMMS study (Hodgen *et al.*, 2009). The second part of this item showed a pentagon with four sides *h* and one side of *t*. This was answered correctly by 50.3% of students in October, rising to 59.5% of students in April (increase not significant). The third part of them item which presents another pentagon was answered correctly by 35.9% in October rising to 50.9% in April. The last part of the item shows an unfinished shape with n sides of length 2. This was answered correctly by only 12.8% of students in October rising to 22.9% in April, which is significant ($p < 0.001$), however the small proportion of students able to correctly answer this task is disappointing. Forming algebraic expressions is a notoriously difficult task, whereby the translation from word problem to expression often results in errors (Bush & Karp, 2013). Another item on simplifying algebraic expressions, was answered correctly by approximately half the sample, 46.8% in October rising to 56.0% in April. There is clear improvement in students dealing with algebraic expressions over the course of the academic year but overall, the proportion able to correctly answer the more difficult tasks is worryingly low. Additionally, evidence of their understanding of a variable as letter evaluated or label only, has emerged. This will be discussed further in section 8.3.4.

8.2.5 Comparing and Ordering Numbers

This item asked students to compare 40% of €400 and 0.75 of €200 by inserting <, > or = in a box between them. Most students did not show any workings with this question with 64.7% and 57.8% answering this item correctly in October and April respectively. The ability to compare and order numbers is required by students in algebra, whereby it allows them to assess if a solution to an equation or inequality is relevant and reasonable (Bottoms, 2003). Difficulty for students often lies in being presented with numbers in different formats such as fractions, decimals, and percentages and for students who answered this item incorrectly this was evident in their workings.

8.2.6 Distributive Property

The item on the distributive property of numbers was answered correctly (or part correctly) by a large proportion of students at both administrations. This basic form of the distributive property, asking students to multiply a number into brackets is often well practised in the form of drill exercises which allows students to build a procedural proficiency dealing with expressions in the form of 4(n-2) (Mok, 2010). However, the correct application does not imply a full understanding of the property and evidence of this emerges with the poor responses to other items on the screener (Mok, 2010). Most students in the sample are fluent in the transformative rules and symbol manipulation required for this item. However, they view these as the superficial movement of symbols without an actual understanding of the structural properties of the distributive law (Kaput, 1989).

The results show that the important content area of equality is well understood by Irish students of this age and that the effort at primary level dedicated to this important concept has been worthwhile (NCCA, 1999). Identification of a linear pattern, forming a simple algebraic expression, comparing and ordering numbers, and the distributive property in simple form are also well understood by the majority of Irish second year students.

8.3 Students' weaknesses in relation to initial algebra

The remaining items on the screener, which assess the content areas of fractions, decimal number magnitude, order of operations, integers, exponents, variables, and equations give rise for concern as they are answered correctly by less than half the sample. Additionally, as mentioned above issues arise with the distributive law when presented in different forms and with forming more difficult algebraic expressions. The results for items answered correctly by less than half of the students are presented in Table 8. The cells highlighted in grey display items that were answered correctly by a smaller proportion of the sample in April when compared with October which indicates a deterioration in understanding of that content.

Table 8: Items on the screener answered correctly by less than half the sample ranked by proportion answered correctly in October

Item Content	Correct or part correct in October	Correct or part correct in April	Not Answered in October	Not Answered in April
Proportional reasoning – students required to explain their reasoning	49.2%	34.4%	6.7%	6.3%
Patterns – asked to identify a statement to best describe the pattern presented	48.5%	60.6%	16.6%	7.5%
Expressions – simplify an algebraic expression	46.8%	59.5%	19.8%	14.0%

Item Content	Correct or part correct in October	Correct or part correct in April	Not Answered in October	Not Answered in April
Equations – identify the next step in solving 5z = 30	45.6%	56.8%	17.3%	8.2%
Fractions – identify an equivalent fraction	42.2%	49.3%	8.6%	5.7%
Fractions – procedural item multiplying fractions	38.4%	45.7%	16.6%	7.8%
Decimal number magnitude	37.3%	35.6%	7.7%	5.2%
Exponents – power raised to a power	35.9%	32.7%	16.8%	9.2%
Expressions – asked to write the expression for a perimeter of a pentagon with 2 equal sides of u and three number value sides.	35.9%	50.9%	24.7%	15.9%
Equations – identify the next step in solving $7h - (3h - 2) = 38$	34.4%	44.2%	18.9%	10.5%
Equation solving 4 - x = 5	31.2%	49.5%	27.9%	15.9%
Order of Operations	29.5%	36.1%	10.8%	6.1%
Patterns – asked to identify an equation for the pattern	28.6%	42.3%	18.9%	9.2%
Variables	22.7%	25.2%	8.6%	5.7%
Equation forming	22.2%	25.4%	15.3%	7.1%
Fractions – relational fraction knowledge	21.1%	30.0%	13.2%	8.0%
Expressions – for an expression for an open shape with equal sides	12.8%	22.9%	38.2%	31.2%
Exponents – asked to square an expression	8.8%	11.9%	19.8%	10.5%

8.3.1 Fractions and Decimals

A cause for concern is the small proportion of students exhibiting a full understanding of fractions at both administrations, given the importance of fraction knowledge in understanding algebra (Booth & Newton, 2014).

Five errors occurred frequently (counts of greater than 100 at the October administration) which demonstrates an overall issue with fraction operations. The first of these errors is when a student applies knowledge of operations on natural numbers to fractions, in this case, dividing is a way to reduce something (Torbeyns *et al.*, 2015). This error was recognised from the item that asked students to identify which of the following expressions show a way to find half a number, *n*? 20% of the sample identified n ÷ 1/2 as a way to do this in October, which reduced to 16% of the sample in April. Another item assessing fractions asked students to select an answer that was equal to 2/3 x x/6. 22% of the sample mistakenly cross multiplied in October with this rising to 24% of the sample in April. A further 20% of students made this error but in addition '*solved*' the expression for 'x' identified as a 'lack of closure' in the literature where the student is not happy to accept an algebraic expression as an answer (Booth, 1988). Another item assessing relational fraction knowledge is known to be difficult for students of this age. Most respondents, 41% and 38% in October and April respectively, chose the option which displays an error of them comparing the denominator of the fraction only. To conclude, the evidence here demonstrates a lack of knowledge of fractions and operations with fractions for Irish second year post-primary students.

A lack of knowledge of decimal number magnitude is evident from the responses to decimal number task on the screener. This item involves the placement of a decimal number on a number line. DeWolf *et al.* (2015) note in correlational studies that the ability to place a decimal number correctly on the number line is one of the best predictors of a student's ability in algebra. Students who place the hatch mark more than 10% away (more than 2 centimetres to the left or right) from the correct location were given an error score to identify their lack of understanding on this item. At the October administration 55% of students received an error score on this item and 59% in April, highlighting the large proportion of students who have difficulties with decimal number magnitude. This warranted further investigation given the importance of this type of understanding to success with algebra overall.

The three different versions of the screener offered three different decimal numbers to place on the number line 0.1529, 0.386, 0.78. Analysis of the proportion of students who placed the hatch mark more than 10% away from the correct location (Error 1A) shows that students found 0.78 the easiest to locate in the correct place. The decimal numbers 0.1529 and 0.386 appeared to create a much greater proportion of errors. Table 9 presents the results for the frequency of errors with the respective decimal numbers. Overall a greater proportion of students make errors when placing decimal numbers in the April administration as indicated in Table 8.

Table 9: Frequency and percentage of students making the decimal number magnitude error

Decimal Number	Frequency of Error 1A at October	% of group	Frequency of Error 1A at April	% of group
0.1529	128	79.5	128	83.1
0.386	141	79.2	115	82.7
0.78	36	20.8	39	24.5

There are different underlying misconceptions that possibly cause these errors. One is the whole number concept, for example, 0.386 is bigger than 0.38 because 386 is bigger than 38. The second possibility is the incorrect application of fraction knowledge where a student assumes that longer decimals are smaller because they contain smaller parts, just as a fraction with a longer denominator contains smaller parts (Durkin & Rittle-Johnson, 2015).

8.3.2 Order of Operations and Integers

Less than 1 in 3 students answered the item assessing order of operations correctly at both administrations (29.5% and 31% at October and April respectively). The most frequent error is students who work from left to right (32.4% and 30% in October and April respectively). A further 15.5% and 18.9% at October and April respectively believe addition supersedes subtraction. This may have to do with a lack of understanding of integers or it may be as a result of having learned order of operations through a mnemonic, for example BIRDMAS (Schwartzman, 1996). The error of adding and subtracting first and then multiplying is lowest. A worrying trend is that a greater proportion of each error occurs in the April administration when compared to the October administration showing errors with the prerequisite content areas increasing slightly as the school year progresses.

No one item on the screener assesses integers specifically, however the error on this item where a student believes addition supersedes subtraction displays a lack of understanding of integers. This item asks the student to evaluate 13 - 3 x 4 + 2 and in doing so the student multiplies first to get 13 - 12 + 2 but then adds 12 and 2 to get 13 - 14 displaying a lack of understanding that 12 is a negative integer, that the minus in essence belongs to the 12. This demonstrates thinking of negative numbers at the "subtrahend level" or the most elementary level, that is a number to be subtracted (Vlassis, 2008). Additional errors with integers are evidenced in other items with equation solving and will be discussed in section 8.3.5 below.

8.3.3 Exponents

Two items on the screener assessed exponents. A large proportion of students at both administrations made errors when working with exponents. Just over one third of students answered the first item correctly in October (35.9%) reducing to less than that proportion (32.7%) in April. This resulted in approximately two in three students making an error when asked to simplify $(x^5)^2$. Exponential notation consists of two parts, a base and exponent and it is a way of expressing repeated multiplication. The ability to understand and work with exponents requires understanding the notation, the meaning and properties of exponents and it is known to be a difficult mathematical topic for students of all ages (Ulusoy, 2019). The most common error was for students to answer that x^{25} is equivalent to $(x^5)^2$, applying the rules of natural numbers and squaring the number 5. This error is made by 27% and 29% of the sample in October and April respectively. Additionally, the students may also have issues with knowledge and understanding of variables given the inclusion of 'x' in the item.

The second item assessing exponents was the worst answered item at both administrations. This item asking students to expand $(x - 2)^2$ assesses exponents, the distributive property, and algebraic expressions. Errors arise as a result of a student confusing squaring an expression with the factorisation of the difference of two squares. The proportion of students making this error increases from the first to the second administration 31.9% to 40.0% (Pinchback, 1991). A lack of understanding of the distributive property often manifests itself when students incorrectly expand expressions such as $(x + 1)^2$ to $x^2 + 1$ (Mok, 2010). The responses to this item present evidence of this lack of understanding with 22.7% and 19.5% of students selecting $x^2 - 4$ as the correct expansion for $(x - 2)^2$ in October and April respectively. A further 14.2% and 15.7% choose $x^2 - 4$ as the correct expansion at each administration.

Findings from a study by Ulusoy (2019) with students in the USA of a similar age showed a lack of understanding of the basic rules with students confusing the base and the exponent at the most basic level, for example or. The findings here show that Irish students in second year struggle with their understanding of exponents and the number of students making these errors increases over the academic year. Further research would be required to investigate what exactly are the issues with exponents. The assessment used by Ulusoy (2019) contains tasks that investigate knowledge of exponents at a more basic level. A separate study using this assessment would be appropriate with Irish second year post-primary students given the evidence that has emerged here in relation to their struggles with exponents.

8.3.4 Variables

Küchemann's (1978) hierarchy of variables should be considered when testing knowledge of variables (Table 2). Given the age of the students and the algebra and number syllabus covered in their first year it is appropriate to start with an assessment task that elicits understanding at the "Letter as Object" or "Letter as Label" level. Students at this age most often believe variables are labels only and it is important to identify this type of incorrect reasoning. The item assessing variables is very similar to one on the PMDT *Maths Competency Test* and is adapted from Küchemann (1981). Students are asked the meaning of the expression $8b + 6m$, where b is the number of books bought and m is the number of magazines bought at a cost of €8 and €6 respectively. A prevalent error emerges from the analysis of this item, where a large proportion of students in the sample view a variable as an object or label (Küchemann, 1978). 39.3% and 45.9% of students answered '8 books and 6 magazines' in October and April respectively. In the original study by Küchemann (1981), it was reported that 39% of 14-year olds made this error. A study from almost twenty years ago in the US used this item (modified to 4 cakes and 3 brownies) and found that 37% of grade 7 students and 27% of grade 8 students made the same error (equivalent to first and second year post primary in Ireland). It appears that despite the changes in pedagogy and approaches to teaching algebra internationally and in Ireland, the proportion of students making this error has not changed much over the decades (Kieran *et al.*, 2016; Prendergast & Treacy, 2017).

Across the sample there is a lack of understanding of variables based on the responses to this item. This lack of understanding of variables may be affecting the performance on other items on the screener for example the item asking students to write an expression for the perimeter of a shape and the items on fractions which include variables. Asquith *et al.* (2007) reported on teachers' knowledge of student understanding and found that often as educators we overestimate students understanding of variables and rarely identify misconceptions students hold about variables. These misconceptions can serve as obstacles to solving problems with variables. There are many consequences when the concept of a variable is not fully understood including mathematics lessons in algebra being rendered incomprehensible for students (Steinle *et al.*, 2009). By identifying our students' misconceptions here namely, misunderstanding a variable as a label, we are better able to plan instruction to address the issue.

The use of patterns to aid understanding and to set the context for the use of variables and expressions can alleviate these misconceptions. As identified in section 8.2.3 students are struggling with identifying an expression to explain a pattern, which may be rooted in their misunderstanding of variables. More research is needed in this specific area. Generational activities are central to algebra and the theme of "generalisation" is to the fore in much of the research into school algebra, including generalising related to patterned activity and generalising related to properties of operations and numerical structure (Kieran *et al.*, 2016). Generalising from numerical and geometric patterns had gained much research attention given students' ability to engage in these tasks (Bourke & Stacey, 1988; Küchemann, 2010; Lee, *et al.*, 2011; Mason, 1996; Warren & Cooper, 2008). This method of engaging learners is known as the functions-based approach to algebra and was introduced to all post-primary schools in Ireland in September 2011 (Prendergast & Treacy, 2017).

8.3.5 Equations

There are several items on the screener that assess knowledge of equations. The first of these asks students to solve $4 - x = 5$. Of note overall from this question is that less than 1 in 3 students could solve the equation $4 - x = 5$ at the October administration although this improves to just under 1 in 2 students correctly solving by April. The proportion of the overall sample not being able to solve the equation was significantly less at the April administration ($p < 0.001$). The main error in attempting to solve the equation arises from the incorrect application of the addition inverse, possibly due to the minus in front of the x, whereby students may be confused as to whether the negative sign means subtraction or is part of a negative number (Vlassis, 2008).

Two items on the screener assess students' ability to identify the next correct step to solve an equation. One item was easier as it assessed the last step of an equation asking for the next step to solve 5z = 30 (Chung & Delacruz, 2014). The most prevalent error (29.5% of students in October and 28.9% in April) saw students incorrectly choosing z = 30 - 5, the incorrect application of subtraction as the inverse to multiplication in solving the equation. The second of these items asked students to identify the next correct step in solving 7h - (3h - 2) = 38. The errors made on this item identify students who do not recognise the minus sign as an operational signifier (Vlassis, 2008). Gallardo and Rojano (1994) give the three main functions of the minus sign: *unary, binary,* and *symmetric*. The unary function relates to the role of the minus sign as attached to a number to form a negative number. The binary function relates to it as an operational sign for arithmetic subtraction and algebraic subtraction (which corresponds to the subtraction of integers or more generally rational numbers). In the symmetric function the minus sign is also viewed as an operational sign consisting of the action of taking the opposite of a number or sum, which is the understanding required in this task (Vlassis, 2008). It should be noted however that the errors on this task may also be due to an incomplete understanding of the distributive property where students have become fluent in the algebraic transformation skills without a true understanding of the distributive law (Mok, 2010).

Solving linear equations is difficult with up to fifty concepts involved and it was not possible to investigate all within the scope of this study (Chung & Delacruz, 2014). However, it is evident that the minus sign is problematic for students where they are possibly viewing it at the subtrahend level of subtraction only (Vlassis, 2008). This is evident in the responses to the item on order of operations, with 15% mistakenly believing that addition supersedes subtraction in October rising to 19% in April. It is also evidenced in the item which asks students to solve 4 - x = 5. In October 14% of students drop or ignore the minus sign when solving this equation and this rises to 18% in April. The exact same proportion of students make an error when asked to simplify an expression combining two negative terms in the expression to get a positive term. Additionally, there is evidence of a lack of true understanding of the distributive property as discussed (Mok, 2010). Furthermore, the role of inverse operations is causing an issue. Where a student showed their workings in solving 4 - x = 5, approximately 10% of these showed incorrect application of the addition inverse at both administrations. Overall, with this item, given the large proportion answering it incorrectly or not at all (68.8% and 50.5% at October and April respectively) it is indicative of issues with applying the correct inverse operation to solve an equation.

The final item assessing the ability to form equations captures the well-known reversal order error. Clement *et al.* (1981) originally posed this problem to university students asking them to write an equation using the variables S and P to represent the following statement: 'There are six times as many students as professors at this university', where S and P represent the number of students and professors respectively. Since then, this problem has been revisited with younger students in many studies (Abouchedid & Nasser, 2000; Fisher, 1988; Philipp, 1992; Swan, 2000). For this study the question swaps the word professors with teachers and the variable *P* with *t*. Unfortunately, there was an increase in the proportion of students making the reversal order error when responding to this item, from 40.9% in October up to 52.2% in April, which is a statistically significant increase ($p = 0.03$). This age group are known to make the reversal order error frequently. For example, a study in Australia found that 91% of all errors made in forming equations were due to the reversal order error (Pawley *et al.*, 2005).

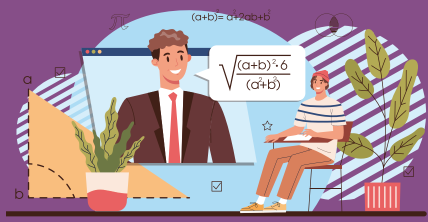
9 Discussion

Algebra can be defined as the language of mathematics and is widely accepted as the gatekeeper to higher education, particularly in the multiple disciplines where mathematics is required (Booth & Newton, 2013; Gavin & Sheffield, 2015; Stacey & Chick, 2004). Algebra has been viewed by many for centuries as the science of equation solving (Kieran, 2004). An ability to work with variables, write algebraic expressions, and to form and solve equations is the essence of success for initial algebra as it is fundamental in preparing for more advanced algebraic concepts (Capraro & Joffrion, 2006). Students are required to have a full understanding of, and an ability to work fluently with, linear equations at junior cycle (NCCA, 2017. However, as identified from the literature many students hold prior misconceptions relating to the prerequisite content areas required for the study of linear equations (Bush & Karp, 2013). To solve a linear equation many underlying concepts are required, and understanding these concepts relies on a solid understanding of many other content areas in mathematics (Chung & Delacruz, 2014). Misconceptions in prerequisite content areas can hinder a student's success in problem solving, prevent the learning of new material and persist despite instruction on the relevant topic (Booth *et al.*, 2015). The literature identifies that errors in the different content areas can increase and decrease across the school year. Specifically errors related to prerequisite content areas such as fractions and mathematical properties should reduce as instruction is delivered during the school year but areas relating to variables, negative signs and equality increase (Booth *et al.*, 2014). The findings in this study are consistent with this – an increase in errors relating to knowledge of variables and exponent items was observed between administrations.

As stated, the items assessing the key content areas of fractions and decimal number magnitude known to predict performance in algebra are poorly answered (Liang *et al.*, 2018). Other areas including order of operation and exponents are also poorly answered. Consequently, the items assessing variables, expressions and equation solving are not well answered aligning with what has been found in previous studies (Booth *et al.*, 2014; Bush & Karp, 2013). These findings suggest that the use of patterns to introduce the concept of variables and functional thinking at junior cycle has not yet reached its potential in supporting students' understanding of a variable as a varying quantity (Project Maths, n.d.). It is imperative that the difficulties students are encountering the prerequisite content areas are addressed through additional instruction and resources. These findings can be used to inform teachers' practice almost immediately. Now that it is known which prerequisite concepts students are struggling with teachers can plan to revise these prior to introducing the algebraic concepts.

Clear evidence previously existed that Irish second year students were struggling with algebra. This evidence was produced from reports based on performance in international assessments and state examinations (Chief Examiner, 2015a; Jeffes *et al.*, 2013). This evidence was retrospective and not specific to actual errors and misconceptions relating to algebra. The empirical results reported here are the first of their kind for Irish second year post-primary students and provide valuable information for mathematics researchers and teachers in the Irish context. There has been major curriculum reform and changes in the approach to the teaching and learning of algebra in Ireland, known as the functions based approach, during the decade preceding this study (Prendergast & Treacy, 2017). Despite this, evidence from this study shows that errors and a lack of understanding of important content areas persist for Irish students. This evidence provides mathematics educators and researchers with much needed information as to the specific areas of weaknesses of our students and instructional intervention and materials can be developed accordingly.

10 Recommendations



Based on the empirical evidence produced in this research it is evident that much work is required to assist students and teachers further. The following recommendations are made based on the results of this research:

- The development of evidence-based teaching interventions (at primary where relevant and post-primary) to target the specific misconceptions. Developing these interventions for online/blended learning would be ideal given the limited resources in terms of time in the Irish classroom.
- Specialist mathematics educators in both primary and post-primary schools to support existing teachers and instruction.
- The development and validation of a suite of cognitive diagnostic assessments in the relevant algebraic content areas for use in the Irish classroom. Cognitive diagnostic assessments combine theories of cognition in a relevant subject domain with advanced statistical scoring models to make inferences about students' strengths and weaknesses (Jang, 2008). They are used to establish a student's level of mastery regarding a defined set of fine-grained cognitive skills (Huang, 2017). These assessments would help educators and researchers fully understand students' misconceptions and thinking in more depth, leading to more targeted use of appropriate interventions (Groß *et al.*, 2016).
- Professional development for mathematics educators where required on how to use the errors identified in this study to highlight and discuss these common errors and misconceptions with their students. Discussion and reflection on these errors during class time could help alleviate these issues for many students (Barbieri & Booth, 2016).
- Offering teachers the opportunity to work collaboratively and discuss findings from their own practice and knowledge of their students to tailor specific plans for their instruction.
- The consideration of these findings at policy level, for both primary and junior cycle levels. It is recommended that a bridging course between primary and post-primary relating to pre-requisite algebra be developed to support learners in the transition from arithmetic to algebraic thinking.

11 Conclusions

Mathematics educators in Ireland have faced huge change in both teaching and assessment over the past decade. The reform can be summarised as consisting of three main changes; what students learn in mathematics, how they learn it and how they are assessed (Prendergast *et al.*, 2017). This research confirms that knowledge of fractions, decimal number magnitude, exponents, integers, order of operations and variables are key content areas where there is a lack of understanding. The importance of this knowledge, which can help mathematics educators guide instruction and highlight required teaching resources and/or interventions to alleviate these issues, is a key outcome of this research. Previous research has shown that when teachers are not aware of their students' prior knowledge then they are inhibited from effectively implementing higher level tasks that introduce new knowledge (Lee *et al.*, 2019). Having empirical evidence to support what teachers are saying might help influence policy makers to provide greater support in providing instructional interventions in these areas.

Encouragingly, the students show the ability to deal with algebraic expressions in a practised manner such as simplifying an expression and multiplying a whole number into brackets using the distributive property. However, their understanding of variables is shown to be at the lower levels. This finding, considering the major reform in the method and mode of teaching algebra, which involved a move away from the procedural and rule-based transformational approach to a deeper conceptual functions-based approach (Prendergast & Treacy, 2017), is disappointing. The students are exhibiting procedural knowledge in terms of dealing with algebraic expressions, however a deeper conceptual knowledge appears to be lacking. This is evidenced clearly in the more complicated tasks of forming algebraic expressions and equations in which only a minority of students were successful. However, this is very much in line with how students perform historically and currently in other nations on these types of tasks (Clement *et al.*, 1981; Dewolf *et al.*, 2016).

While procedural knowledge, the how of something, is very important for mathematical proficiency, it is conceptual knowledge, the why of something, that supports a full understanding and leads to true proficiency (Booth & Davenport, 2013). Both types of knowledge are essential for students' overall success with a subject (Ross & Willson, 2012). The major reform and introduction of the new mathematics syllabus in 2010 focused on conceptual understanding where students are expected to engage in problem solving to deepen their understanding of topics (Prendergast & Treacy, 2017). Based on the findings of this research, this shift in the focus and methods of teaching algebra has yet to reach its potential in improving students' conceptual understanding. However, as noted it is early in the implementation of the reform and the results must be viewed in this light. The results are very much in line with what was reported by the Chief Examiner based on the Junior Certificate examination of 2015.

Irish students struggle with initial algebra as much as their international counterparts. Evidence presented here shows common errors and misconceptions recognised in the literature and provides a much greater depth of understanding of Irish post-primary students' difficulties than was previously known. It is hoped that this research will serve all stakeholders in mathematics education in understanding the strengths and weaknesses of students as they begin to grasp algebraic concepts. The information produced here can be used to determine how best to support teachers and students in this area of mathematics education which is well documented as difficult to teach and learn (Demonty *et al.*, 2018; Kieran, 2007).

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